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On statistical and deterministic quantum teleportation

Vladimir Božin†‡, Marina Hruška†, Nikola Paunković† and Fedor Herbut†§

† Faculty of Physics, University of Belgrade, PO Box 368, 11001 Beograd, Yugoslavia

‡ Faculty of Mathematics, University of Belgrade, Studentski TRG 12, Beograd, Yugoslavia

§ Serbian Academy of Sciences and Arts, Knez Mihajlova 35, Beograd, Yugoslavia

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Abstract. We generalize quantum teleportation to, what we call, statistical teleportation utilizing previous results on distant preparation, and on the basic ingredient entities of an entangled composite-system state vector. Our main result is 'the central theorem', establishing a simple necessary and sufficient condition for the crucial entity: the event that the sender of a pure quantum state has to measure in the first step of the two-step (and two-laboratory) teleportation procedure. We derive numerous consequences especially for deterministic teleportation (a special case of statistical teleportation), which is a direct generalization of the known quantum teleportation. Detailed further generalization to proper and improper mixtures is investigated. Finally, it is shown that extension to teleportation with nonlinear distant preparation is not possible unless the idea of teleportation is essentially changed.

1. Introduction

Entangled composite-system state vectors are known to display nonclassical statistical correlations between the subsystems (so-called nonseparability (D'Espagnat 1976)). A major breakthrough in the physical understanding of nonseparability was made by Schrödinger (1936) when he discovered distant preparation and when he obtained the first results on its amazing scope.

Entanglement theory, also known as distant correlations theory, was further developed in great detail and generality along the lines of Schrödinger's approach by one of us (FH) and Vujičić (1976) (Vujičić and Herbut 1984, 1988). It was done in the framework of antilinear Hilbert–Schmidt operators (which have not found sufficient application in the further literature on distant correlations) and distant measurement (which is not particularly relevant for teleportation). For the readers' convenience, most of the previous results on distant correlations that we find relevant for teleportation are restated in this article in a new and simplified way. Since they are rather elementary and not surprising, the proofs are omitted.

In understanding entanglement, the recent discovery of teleportation by Bennett *et al* (1993) seems to be one of the major events. It is receiving continued attention (e.g. Popescu 1994, Bennett 1995, Bennett *et al* 1996, Nielsen and Caves 1997, Moussa 1997 etc).

Investigating how wide the choice of the entities used in quantum teleportation is, i.e. what features these entities should have in order to make quantum teleportation possible, we find it appropriate to introduce the concept of *statistical teleportation* (see section 3). It corresponds to a particular event that is measured as an observable in the first step. Its occurrence (result 1) takes place with a probability, or, equivalently, in a subensemble. It is also the basic building block of deterministic teleportation (see (9) below).

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Statistical teleportation is a two-step and two-laboratory phenomenon. The first step takes place in the nearby laboratory in the way of a measurement that via *distant preparation* of a suitable state vector, and a classical message, brings about a change in the distant laboratory. The second step consists of the application of a suitable unitary operation in the distant laboratory converting the distantly prepared state vector into the teleported one.

Investigation of the first step of statistical teleportation is not only an application, but also a further development of distant preparation theory: it is elaboration of the special case when distant preparation is a linear map.

Some attempts at practical application of deterministic or perfect teleportation actually fall back on statistical teleportation and achieve only a 25%, or at best a 50%, absolute efficiency (Braunstein and Mann 1995). Hence, in applications, statistical teleportation can be expected to play an important role both as a constituent of perfect, or of partially realized or imperfect, teleportation or as an outright competitor to perfect teleportation.

In order to gain a balanced view about the potential importance of the concept of statistical teleportation that is being proposed, one should also be aware of the fact that recently, in the context of the notion of interaction-free measurement, the idea of statistical measurement appeared in the literature (Elitzur and Vaidman 1993, Kwiat *et al* 1995, 1996). Namely, in individual cases the measurement at issue may or may not succeed; it takes place with a probability. Ensemblewise speaking, we have measurement in a subensemble, just like the case with teleportation when it is statistical.

A summary of distant-preparation theory is presented in section 2. In section 3 the concept of statistical teleportation is defined and its relation to deterministic teleportation is clarified. A summary on the basic ingredients of a composite-system state vector is given in section 4. The central result on statistical teleportation is derived in section 5. Its immediate consequences are derived in section 6. Section 7 is devoted to a detailed discussion of deterministic teleportation. Generalization to proper and improper mixtures is performed in sections 8 and 9 respectively. Finally, concluding remarks are given in section 10.

Since we shall be dealing with entities containing redundancies, we shall extract the relevant parts (relevant entities) from them. Whenever possible we utilize a simpler notation (and simpler term) for the relevant entity. In particular, the redundant entities will be denoted with bars, and the relevant ones without them.

2. Partial scalar product and distant preparation

In this section we restate some relevant basic results on distant preparation developed in the previously mentioned articles.

Distant preparation requires an entangled state vector of a composite system consisting of two subsystems. It comes about when one chooses one of the subsystems to perform direct subsystem measurement on it. We call the directly measured subsystem the nearby one, and we denote it by 'n'. In distant preparation one studies the consequences of the direct subsystem measurement on the state of the other subsystem. We call the latter the distant subsystem and we denote it by 'd'.

To obtain the mentioned consequences, it is important that, during the direct measurement on the nearby subsystem, neither the measuring apparatus nor this subsystem must interact with the distant subsystem. The latter changes its quantum-mechanical state due to the very procedure described, and this is called *distant preparation*. Most often the subsystems are far apart or distant from each other so that the direct subsystem measurement on the nearby subsystem can be easily performed without influencing dynamically the state of the distant subsystem.

We denote the state spaces (Hilbert spaces) of the nearby and the distant subsystems by $\mathcal{H}_n$ and $\mathcal{H}_d$ respectively.

Lemma 1. Let $|\chi\rangle_{nd}$ $(\in (\mathcal{H}_n \otimes \mathcal{H}_d))$ and $|\phi\rangle_n$ $(\in \mathcal{H}_n)$ be two nonzero vectors. Let $|\chi\rangle_{nd} = \sum_{i=1}^{I} \alpha_i |i\rangle_n |i\rangle_d$ ('I' finite or $\infty, \forall i : \alpha_i \in \boldsymbol{C}$) be an expansion of $|\chi\rangle_{nd}$ in terms of uncorrelated vectors. Then the vector

$$\sum_{i=1}^{I} (\alpha_i \langle\phi|_n |i\rangle_n) |i\rangle_d$$

whereby $(\langle\phi|_n|i\rangle_n)$ we denote the number obtained by ordinary scalar multiplication in $\mathcal{H}_n$, is an element of $\mathcal{H}_d$, and it is one and the same whatever the mentioned expansion of $|\chi\rangle_{nd}$ in uncorrelated vectors.

Definition 1. The so-called *partial scalar product*, denoted by $\langle\phi|_n|\chi\rangle_{nd}$ and belonging to $\mathcal{H}_d$, is evaluated by expanding $|\chi\rangle_{nd}$ in terms of uncorrelated vectors in an arbitrary way (cf lemma 1), and by using the ordinary scalar product in $\mathcal{H}_n$. If we have, for example $|\chi\rangle_{nd} = \sum_{i=1}^{I} \alpha_i |i\rangle_n |i\rangle_d$, then, by definition,

$$\langle\phi|_n|\chi\rangle_{nd} \equiv \sum_{i=1}^{I} (\alpha_i \langle\phi|_n |i\rangle_n) |i\rangle_d. \tag{1}$$

Corollary 1. One has

$$\langle\lambda|_d (\langle\phi|_n|\chi\rangle_{nd}) = (\langle\phi|_n \langle\lambda|_d) |\chi\rangle_{nd}.$$

(Note that the LHS is a scalar product in $\mathcal{H}_d$, and the RHS is one in $(\mathcal{H}_n \otimes \mathcal{H}_d)$.)

Corollary 2. Partial scalar multiplication is continuous and linear in the composite-system vector.

Corollary 3. If A_n is a bounded Hermitian operator in $\mathcal{H}_n$, then we have the following equality of partial scalar products

$$\langle\phi|_n ((A_n \otimes I_d)|\chi\rangle_{nd}) = (\langle\phi|_n A_n)|\chi\rangle_{nd}$$

where I_d is the identity operator in $\mathcal{H}_d$.

Corollary 4. If A_d is a bounded linear operator in $\mathcal{H}_d$, then

$$A_d(\langle\phi|_n|\chi\rangle_{nd}) = \langle\phi|_n((I_n \otimes A_d)|\chi\rangle_{nd})$$

always, where I_n is the identity operator in $\mathcal{H}_n$.

Corollary 5. Let $|\phi\rangle_n \in \mathcal{H}_n$, $\langle\phi|_n|\phi\rangle_n = 1$. Then

$$(|\phi\rangle_n\langle\phi|_n \otimes I_d)|\chi\rangle_{nd} = |\phi\rangle_n \otimes (\langle\phi|_n|\chi\rangle_{nd}). \tag{2}$$

In this outline we deal with *distant preparation* under two restrictions. We assume that the composite system is in a pure state, described by a state vector $|\chi\rangle_{nd}$ for example and that it is an elementary event (or atom), $|\phi\rangle_n\langle\phi|_n$ for example, that occurs or not on the nearby subsystem.

If the observable (projector) $|\phi\rangle_n\langle\phi|_n$ (or $(|\phi\rangle_n\langle\phi|_n \otimes I_d)$) is measured and if one obtains 1 as the result of the measurement, then, one speaks of the *occurrence of the event* at issue (see Gudder 1979). The result 0 is referred to as nonoccurrence.

Switching over from the more usual individual-system language to the (equivalent) ensemble one, one can say that after the measurement of $(|\phi\rangle_n\langle\phi|_n \otimes I_d)$ in $|\chi\rangle_{nd}$ we leave out all those distant subsystems on the nearby partners of which $|\phi\rangle_n\langle\phi|_n$ has not occurred.

The remaining subensemble of distant subsystems, on the nearby partners of which $|\phi\rangle_n\langle\phi|_n$ has occurred, is, by definition, *the distantly prepared subensemble* (or state).

The fraction of distant subsystems that goes into the distantly prepared subensemble is obviously equal to the probability of the occurrence of $(|\phi\rangle_n\langle\phi|_n \otimes I_d)$ in $|\chi\rangle_{nd}$.

The distantly prepared state can be easily evaluated if one confines oneself to ideal measurement. However, before we state this result, we introduce the concept and evaluation of the state operator of a subsystem.

Definition 2. If a composite system is in the state given by the state vector $|\chi\rangle_{nd}$, then the state of the distant subsystem is given by the reduced statistical operator (state operator)

$$\rho_d \equiv \mathrm{Tr}_n\, |\chi\rangle_{nd}\langle\chi|_{nd} \equiv \sum_i (\langle\phi_i|_n|\chi\rangle_{nd})(\langle\chi|_{nd}|\phi_i\rangle_n) \tag{3}$$

where Tr_n denotes the partial trace in $\mathcal{H}_n$, and $\{|\phi_i\rangle_n : \forall i\}$ is a complete orthonormal (ON) basis in $\mathcal{H}_n$. The definition and evaluation of the nearby-subsystem state ρ_n is symmetrical to (3) with respect to interchange of n and d.

Taking the matrix representation of ρ_d in a complete ON basis in $\mathcal{H}_d$, the RHS of (3) is, due to corollary 1, seen to reduce to the usual definition of the partial trace, which is known to be independent both of the choice of the basis $\{|\phi_i\rangle_n : \forall i\}$ in $\mathcal{H}_n$ (cf (3)) and the mentioned basis in $\mathcal{H}_d$.

Corollary 6. For any bounded Hermitian operators A_d and B_d one has

$$A_d[\mathrm{Tr}_n(|\chi\rangle_{nd}\langle\chi|_{nd})]B_d = \mathrm{Tr}_n[(I_n \otimes A_d)(|\chi\rangle_{nd}\langle\chi|_{nd})(I_n \otimes B_d)].$$

And symmetrically, for any bounded Hermitian operators A_n and B_n:

$$A_n[\mathrm{Tr}_d(|\chi\rangle_{nd}\langle\chi|_{nd})]B_n = \mathrm{Tr}_d[(A_n \otimes I_d)(|\chi\rangle_{nd}\langle\chi|_{nd})(B_n \otimes I_d)].$$

Proposition 1 (distant preparation by ideal subsystem measurement). Let $|\phi\rangle_n$ $(\in \mathcal{H}_n)$ and $|\chi\rangle_{nd}$ $(\in (\mathcal{H}_n \otimes \mathcal{H}_d))$ be state vectors, and let us write

$$\langle\phi|_n|\chi\rangle_{nd} = w^{1/2}|\omega\rangle_d \tag{4}$$

where w is by definition the square of the norm of the vector on the LHS. If the event $(|\phi\rangle_n\langle\phi|_n \otimes I_d)$ occurs in *ideal* nearby-subsystem measurement in the state $|\chi\rangle_{nd}$ of the composite system, then the probability of the occurrence is w, and the distantly prepared state vector is $|\omega\rangle_d$.

Proof. The required probability p is, as well known,

$$p = \langle\chi|_{nd}(|\phi\rangle_n\langle\phi|_n \otimes I_d)|\chi\rangle_{nd}$$

which, utilizing idempotency and (2) twice, gives

$$p = \|\langle\phi|_n|\chi\rangle_{nd}\|^2 = w$$

as claimed.

Since in ideal measurement the change of state due to occurrence is described by the Lüders formula (Messiah 1961, Lüders 1951), the occurrence of $(|\phi\rangle_n\langle\phi|_n \otimes I_d)$ in $|\chi\rangle_{nd}$ changes this composite-system state vector into the state vector

$$|\bar{\chi}\rangle_{nd} \equiv w^{-1/2}(|\phi\rangle_n\langle\phi|_n \otimes I_d)|\chi\rangle_{nd}.$$

The distant subsystem is, then, described by the reduced statistical operator (cf (3))

$$\bar{\rho}_d \equiv \mathrm{Tr}_n\, |\bar{\chi}\rangle_{nd}\langle\bar{\chi}|_{nd}$$

which, after substitution of the preceding relation, using (2), and finally of (4), becomes

$$\bar{\rho}_d = \mathrm{Tr}_n(|\phi\rangle_n\langle\phi|_n \otimes |\omega\rangle_d\langle\omega|_d) = |\omega\rangle_d\langle\omega|_d$$

as claimed. □

As is well known, three requirements are relevant for individual-system measurements. (i) In each measurement a result (characteristic value) is obtained, and their relative frequencies, in the limit of infinitely many measurements, reproduce the probability values predicted by quantum mechanics. (ii) The results are repeatable, i.e. an immediately repeated measurement of the same observable on the same individual system gives the same result with certainty. (iii) Possible results on compatible observables are preserved in each individual-system measurement.

If all three requirements are satisfied, we have ideal measurement (cf Herbut 1974, theorem 1). If the third is not, but the first two are, then we are dealing with nonideal first-kind or repeatable measurements (Pauli 1933, Busch *et al* 1991). Finally, if only requirement (i) is valid, one has second-kind or nonrepeatable measurement (ibid).

If one has in mind nonideal first-kind or second-kind measurement, then the probability is the same as in ideal measurement because it does not depend on the kind of measurement performed. But, one wonders if a similar statement is true about the distantly prepared subensemble: is it still describable by a state operator? If yes, can this be evaluated? An answer to the latter question may be burdened by the fact that there is no change-of-state formula in nonideal measurement.

Our chance lies in the fact that the concept of 'state' serves to provide us with probabilities, and it is determined by the totality of the latter. Hence, we have to convert our change-of-state problem into the language of mere probabilities.

In an attempt to generalize distant preparation from ideal to *any kind of individual-system measurement*, it seems reasonable to base our further considerations on the following *two physical assumptions*.

(i) The probability of occurrence of two compatible events in immediate succession (i.e. when the time interval between the occurrences tends to zero) equals the probability of coincidence, i.e. of joint measurement of the two compatible events. And this is valid for any kind of measurement.

(ii) If $|\lambda\rangle_d$ is an arbitrary state vector in $\mathcal{H}_d$, there exists a probability $p(\lambda)$ for the occurrence of $|\lambda\rangle_d\langle\lambda|_d$ in the distantly prepared state by any measurement. (This probability is conditional by the very definition of the distantly prepared state.)

Now, we can prove the basic relevant result of this section.

Theorem 1. Let $|\phi\rangle_n$ and $|\chi\rangle_{nd}$ be state vectors. We assume that in an arbitrary measurement of the event $(|\phi\rangle_n\langle\phi|_n \otimes I_d)$ in the state $|\chi\rangle_{nd}$ the occurrence has a positive probability w. The occurrence brings *ipso facto* the distant subsystem into the state $|\omega\rangle_d$ defined by (4). Besides, also the probability w is given by (4) (just like in ideal measurement).

Proof. We apply the above two assumptions to the immediate succession of the events $(|\phi\rangle_n\langle\phi|_n \otimes I_d)$ and $(I_n \otimes |\lambda\rangle_d\langle\lambda|_d)$ in the state $|\chi\rangle_{nd}$:

$$wp(\lambda) = \langle\chi|_{nd}(|\phi\rangle_n\langle\phi|_n \otimes |\lambda\rangle_d\langle\lambda|_d)|\chi_{nd}\rangle. \tag{5}$$

(Note that on the LHS we have the usual conditional-probability factorization.)

The probability $p(\lambda)$ is determined by the rest of the entities in (5), and all these, being probabilities, are independent of the kind of measurement. Hence, the probability $p(\lambda)$ is in any measurement the same as in ideal measurement. In the latter we know (see proposition 1) that $p(\lambda) = |\langle\omega|_d|\lambda\rangle_d|^2$, i.e. that the distantly prepared state is described by $\rho_d = |\omega\rangle_d\langle\omega|_d$ (cf (4)). Hence, this is true for all measurements. □

We have thus been able to generalize the physical meaning of $|\omega\rangle_d$ defined in (4) from ideal to any kind of measurement on account of the above physical assumptions. Bell and Nauenberg (1966) have made assumption (i), but only for ideal measurement (on a noncomposite system), and they have derived the Lüders formula (Messiah 1961, Lüders 1951). We have extended their assumption to all kinds of measurement. This seems justified because probabilities in quantum mechanics do not depend on the kind of measurement performed.

3. Definition of statistical teleportation

The investigation that follows will be based on relation (4) evaluating the distantly prepared state vector. Let us introduce the concepts required for a definition of statistical teleportation.

The system on which quantum teleportation takes place consists of three subsystems. We denote them by 1, 2, and 3. There are two laboratories. The experimenter in the nearby one, who is the sender of a quantum-state vector, has the nearby subsystem $(1+2)$ at his disposal, whereas the experimenter in the distant laboratory, who is the receiver, is able to act on the state of the distant subsystem 3.

The most important entity for teleportation is a given entangled state vector $|\bar{\Psi}\rangle_{23}$ of the composite system $(2+3)$. It plays the role of a sort of a bridge or a specific quantum information channel because, as stated, subsystem 2 is in the nearby laboratory, and subsystem 3 is in the distant one. We call $|\bar{\Psi}\rangle_{23}$ *the bridge state vector*.

In the nearby laboratory one measures a suitable observable, an elementary event (synonym: an atom) $|\bar{a}\rangle_{12}\langle\bar{a}|_{12}$, which is determined by a state vector $|\bar{a}\rangle_{12}$ that we call the *atomic state vector*. This measurement is actually performed in the three-subsystem composite-system state described by $|\psi\rangle_1|\bar{\Psi}\rangle_{23}$, where $|\psi\rangle_1$ is a state vector for subsystem 1 that is to be teleported. As a consequence of the occurrence of the event $(|\bar{a}\rangle_{12}\langle\bar{a}|_{12} \otimes I_3)$ in its measurement (I_3 being the identity operator in $\mathcal{H}_3$), that occurs with some positive probability $\bar{w}$, a distantly prepared state $|\psi\rangle'_3$ appears in the distant laboratory. It is evaluated by taking the partial scalar product

$$\langle\bar{a}|_{12}|\psi\rangle_1|\bar{\Psi}\rangle_{23} = (\bar{w})^{1/2}|\psi\rangle'_3 \qquad (6a)$$

(see proposition 1 and theorem 1).

The experimenter in the distant laboratory has an operation $\bar{\mathcal{U}}_3$ at his disposal, which is suitably chosen for (or associated with) the atomic state vector $|\bar{a}\rangle_{12}$. If the crucial event $(|\bar{a}\rangle_{12}\langle\bar{a}|_{12} \otimes I_3)$ does occur, a classical message (a phone call, for example) goes from the nearby laboratory to the distant one. Then in the latter the mentioned operation $\bar{\mathcal{U}}_3$ is applied to the distantly prepared state $|\psi\rangle'_3$, and it is thus converted into the teleported state $|\psi\rangle_3$.

More precisely, let $\boldsymbol{S}_1$ be some linear manifold of first-subsystem states. Then, for each $|\psi\rangle_1 \in \boldsymbol{S}_1$,

$$|\psi\rangle_3 = \bar{\mathcal{I}}_{31}|\psi\rangle_1 \qquad (7)$$

where '$\bar{\mathcal{I}}_{31}$' is the well known unitary isomorphism mapping $\mathcal{H}_1$ onto $\mathcal{H}_3$ and giving *physically the same* state, or it is some other fixed isomorphism of $\mathcal{H}_1$ onto $\mathcal{H}_3$. (If the latter is the case, then we have some kind of generalized teleportation, but the theory is formally the same). If $\bar{\mathcal{I}}_{31}$ is the 'same-state' isomorphism, then subsystems 1 and 3 have to be of the same physical nature, of course. If $\bar{\mathcal{I}}_{31}$ is more general, $\mathcal{H}_1$ and $\mathcal{H}_3$ have to be only isomorphic. We denote by '$\mathcal{I}_{31}$' the map obtained by restricting the domain of $\bar{\mathcal{I}}_{31}$ to $\boldsymbol{S}_1$.

One can make the following summary.

Definition 3. We call a two-step procedure in two distantly separated laboratories that is performed with the purpose to transmit an arbitrary state vector $|\psi\rangle_1$ from some linear manifold $\boldsymbol{S}_1$ in $\mathcal{H}_1$ into the physically identical (or isomorphic) state vector $|\psi\rangle_3$ in $\mathcal{H}_3$ *statistical teleportation* if the following requirements are satisfied:

(i) The first step is distant preparation, we denote it by $\mathcal{U}_{31}$, taking state vectors from $\boldsymbol{S}_1$ into state vectors in $\mathcal{H}_3$. It is performed in the nearby laboratory by measuring a suitable observable, an event of the form $(|\bar{a}\rangle_{12}\langle\bar{a}|_{12} \otimes I_3)$, $|\bar{a}\rangle_{12} \in (\mathcal{H}_1 \otimes \mathcal{H}_2)$, in the state $|\Psi\rangle_1|\bar{\Psi}\rangle_{23}$, where $|\bar{\Psi}\rangle_{23}$ is a given entangled state vector in $(\mathcal{H}_2 \otimes \mathcal{H}_3)$, shared between the two laboratories (subsystem 2 being in the nearby laboratory and subsystem 3 in the distant laboratory). The suitable chosen state vector $|\bar{a}\rangle_{12}$ is called atomic.

(ii) Information about occurrence or nonoccurrence of the mentioned event is transmitted by a classical channel from the nearby to the distant laboratory. In case of occurrence, the second step of teleportation, a unitary operation in $\mathcal{H}_3$, transforms the distantly prepared state of subsystem 3 into the physically identical (or isomorphic) teleported state vector.

We denote the linear manifold spanned by the range of $\mathcal{U}_{31}$, i.e. $\mathcal{U}_{31}(\boldsymbol{S}_1)$ by $\boldsymbol{S}_3'$, the restriction of operation $\bar{\mathcal{U}}_3$ to $\boldsymbol{S}_3$ by $\mathcal{U}_3$, and, finally, the linear manifold $\mathcal{U}_3(\boldsymbol{S}_3')$ by $\boldsymbol{S}_3$.

The two-step and two-laboratory procedure at issue is described by the following chain of operations

$$\mathcal{I}_{31} = \mathcal{U}_3 \circ \mathcal{U}_{31} \tag{8}$$

where '$\circ$' denotes 'after' reading from left to right, as displayed on the commuting figure 1. (So far (8) is confined to state vectors in $\boldsymbol{S}_1$.)

The term 'statistical teleportation' is suggested having an individual (1+2+3) composite system in mind. One may also use the synonym 'teleportation with a probability'. If one envisages an ensemble of (1 + 2 + 3) systems of the kind described, one may use the synonym 'teleportation in a subensemble'.

We now *require* $\mathcal{U}_3$ to be *linear*, which, in view of the fact that it preserves the norm, is equivalent to requiring $\mathcal{U}_3$ to be a unitary isomorphism mapping $\boldsymbol{S}_3'$ onto $\boldsymbol{S}_3$. On account of relation (8), also $\mathcal{U}_{31}$ is then linear and hence a unitary isomorphism taking $\boldsymbol{S}_1$ onto $\boldsymbol{S}_3'$. (This is where elaboration of distant preparation $\mathcal{U}_{31}$ comes into play.)

The above requirement of linearity of $\mathcal{U}_3$ (and of $\mathcal{U}_{31}$) is crucial for our theory. At the end of this article (cf section 10.1) this requirement is investigated, and it is proved that it is a necessary consequence of some very basic assumptions.

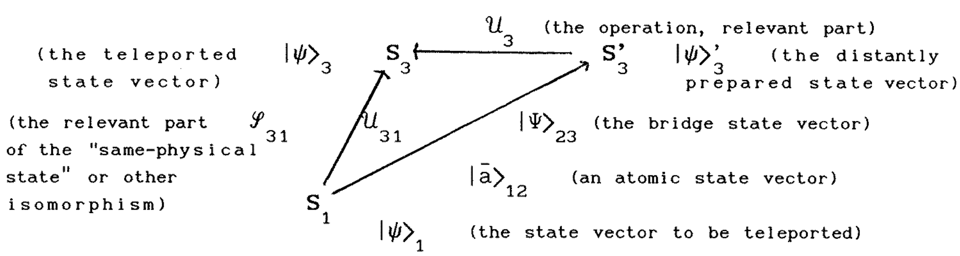

Figure 1.

Evidently, the distant experimenter must interact very nontrivially with subsystem 3 to bring about the action of $\mathcal{U}_3$. This interaction, whatever its form, is a part of the spontaneous evolution of the distant laboratory. Unitary evolution is confined, as well known, to dynamically isolated systems, and subsystem 3 is not at all isolated.

A way out of this apparent paradox lies in the fact that the instruments with which the distant experimenter acts on subsystem 3 are macroscopic bodies. Thus, they are tremendously large compared to subsystem 3, which is thought of as a microscopic system. Hence, the feedback (or reaction) on the instruments can be neglected in a very good approximation. This makes subsystem 3 'move in an external potential'.

We have seen that statistical teleportation is an intricate process that has distant preparation at its heart. *Deterministic* (or perfect) *teleportation*, invented by Bennett *et al* (1993), is slightly more complex than statistical teleportation, and the latter is the essential ingredient of the former.

Namely, one has a complete observable in $\mathcal{H}_{12}$ (i.e. one with all characteristic values nondegenerate) all characteristic vectors of which are atomic state vectors:

$$A_{12} \equiv \sum_{m=1}^{M} a_m |\bar{a}^{(m)}\rangle_{12} \langle \bar{a}^{(m)}|_{12} \qquad (m \neq m' \Rightarrow a_m \neq a_{m'}) \tag{9}$$

where $M \equiv \dim(\mathcal{H}_1) \times \dim(\mathcal{H}_2)$.

The distant experimenter has an operation $\bar{\mathcal{U}}_3^{(m)}$ associated with each of the atomic state vectors $|\bar{a}^{(m)}\rangle_{12}$ at his disposal. After the measurement of $(A_{12} \otimes I_3)$ in the state $|\psi\rangle_1 |\bar{\Psi}\rangle_{23}$ a 'phone call' informs the distant experimenter which result a_m has occurred. He applies the corresponding operation $\bar{\mathcal{U}}_3^{(m)}$, and thus he necessarily obtains one and the same teleported state $|\psi\rangle_3$ ($\equiv \mathcal{I}_{31}|\psi\rangle_1$) whatever the classical message. This is why we speak of 'deterministic' teleportation, or teleportation with certainty, in this case. (The ensemble point of view is unnecessary here.)

It is our first aim to find all atomic state vectors $|\bar{a}\rangle_{12}$, i.e. state vectors which make statistical teleportation possible when a bridge state vector $|\bar{\Psi}\rangle_{23}$ and a linear manifold $\boldsymbol{S}'_1 (\subseteq \mathcal{H}_1)$ are given. The operation $\mathcal{U}_3$ is, then, determined by $\mathcal{U}_{31}$ (cf (8)), which is, in turn, determined by $|\bar{a}\rangle_{12}$ via (6*a*), rewritten as

$$\mathcal{U}_{31}|\psi\rangle_1 = |\psi\rangle'_3. \tag{6b}$$

But before we proceed with our theory of statistical teleportation, we expound further the 'tools' of distant preparation theory because we need a precise statement on the scope of distant preparation (to see how wide the linear manifold $\boldsymbol{S}_3$ can turn out to be).

4. The basic ingredient entities of a composite-system state vector

In this section we restate the relevant part of a distant correlations theory from previous work (Herbut and Vujičić 1976, Vujičić and Herbut 1984 and 1988). We resume the notation of section 2 calling the two subsystems the 'nearby' and the 'distant' one.

The subsystem state operators and the correlation operator, the basic ingredient entities of a composite-system state vector $|\chi\rangle_{nd}$ ($\in (\mathcal{H}_n \otimes \mathcal{H}_d)$), are the tools in terms of which our theory of teleportation is going to be developed.

Lemma 2. Let $|\chi\rangle_{nd}$ be a composite-system state vector, and $\{|\phi_k\rangle_n : \forall k\}$ a complete ON basis in $\mathcal{H}_n$. Then the following claims are valid.

(i) There exists a set of vectors $\{|\omega'_k\rangle_d : \forall k\}$ ($\subseteq \mathcal{H}_d$) such that

$$|\chi\rangle_{nd} = \sum_k |\phi_k\rangle_n |\omega'_k\rangle_d. \tag{10a}$$

(ii) The vectors $|\omega'_k\rangle_d$, called generalized expansion coefficients, are unique and each depends only on the corresponding subsystem basis vector $|\phi_k\rangle_n$ and on $|\chi\rangle_{nd}$.

(iii) The generalized expansion coefficients can be evaluated as partial scalar products:

$$\forall k: \qquad |\omega'_k\rangle_d = \langle\phi_k|_n|\chi\rangle_{nd}. \tag{10b}$$

The subsystem state operators (reduced statistical operators) ρ_n and ρ_d of a given composite-system state vector $|\chi\rangle_{nd}$ were defined in definition 2.

Lemma 3. A generalized expansion coefficient $|\omega'_k\rangle_d$ in (10*a*) is zero if and only if the corresponding vector $|\phi_k\rangle_n$ belongs to the null space of ρ_n. Equivalently, one has an expansion in a subsystem ON subbasis $\{|\phi_k\rangle_n : \forall k\}$ of the form (10*a*) if and only if the subspace spanned by this subbasis contains the range of ρ_n.

Corollary 7. If a subsystem ON subbasis $\{|\phi_k\rangle_n : \forall k\}$ such that $|\chi\rangle_{nd}$ can be expanded in it (cf lemma 3) is given, and in the corresponding generalized expansion coefficients $|\omega'_k\rangle_d$ in (10*a*) the norms '$w_k^{1/2}$' are explicitly displayed

$$\forall k: |\omega'_k\rangle_d \equiv w_k^{1/2}|\omega_k\rangle_d \tag{11a}$$

then one has the following (mathematical) decomposition of the state operator ρ_d of the distant subsystem into pure states:

$$\rho_d = \sum_k w_k|\omega_k\rangle_d\langle\omega_k|_d. \tag{11b}$$

From the physical point of view, decomposition (11*b*) is a potential, not an actual, one because, taking the ensemble point of view, ρ_d cannot be actually thought of as decomposed into distinct subensembles (corresponding to the terms in (11*b*)) in view of the homogeneity of the ensemble $|\chi\rangle_{nd}\langle\chi|_{nd}$ of composite systems. But the (nonselective) measurement of a complete nearby-subsystem observable

$$(B_n \otimes I_d) = \sum_k b_k(|\phi_k\rangle_n\langle\phi_k|_n \otimes I_d) \qquad k \neq k' \Rightarrow b_k \neq b_{k'}$$

(the $|\phi_k\rangle_n$ same as in lemma 3) in the state $|\chi\rangle_{nd}$ brings about (11*b*) as an actual decomposition. This is an immediate consequence of theorem 1. It is called *distant state operator decomposition* (or empirically: *distant ensemble decomposition*). It is the nonselective counterpart of distant preparation. (The latter is a notion connected with the selective measurement of one characteristic value b_k of $(B_n \otimes I_d)$.)

Lemma 4. Expansion (10*a*) in a subsystem ON subbasis is biorthogonal, i.e. also the set $\{|\omega'_k\rangle_d : \forall k\}$ of the generalized expansion coefficients is a set of orthogonal vectors, if and only if $\{|\phi_k\rangle_n : \forall k\}$ is a characteristic subbasis of ρ_n.

Definition 4. If one writes a biorthogonal expansion of $|\chi\rangle_{nd}$ in terms of ON subbases $\{|\phi_i\rangle_n : \forall i\}$ and $\{|\omega_i\rangle_d : \forall i\}$ for the two subsystems

$$|\chi\rangle_{nd} = \sum_i r_i^{1/2}|\phi_i\rangle_n|\omega_i\rangle_d \tag{12}$$

with positive $r_i^{1/2}$, then one says that a *Schmidt biorthogonal expansion* of $|\chi\rangle_{nd}$ is given. The vectors $(r_i^{1/2}|\omega_i\rangle_d)$ are the generalized expansion coefficients in (10*a*), and $r_i^{1/2}$ are their norms (cf (11*a*)).

It is obvious from lemma 4 and definition 4 that every composite-system state vector $|\chi\rangle_{nd}$ can be written as a Schmidt biorthogonal expansion, and that the latter is, in general, not unique as far as the basis vectors are concerned.

Corollary 8. If a Schmidt biorthogonal expansion (12) is given, one can read off spectral forms of ρ_n and ρ_d. They are given by the relations:

$$\rho_n = \sum_i r_i |\phi_i\rangle_n \langle\phi_i|_n \qquad \forall i : r_i > 0 \tag{13a}$$

$$\rho_d = \sum_i r_i |\omega_i\rangle_d \langle\omega_i|_d \qquad \forall i : r_i > 0. \tag{13b}$$

(Notice that the positive characteristic values of ρ_n and ρ_d necessarily coincide.)

Remark 1. In view of (13*a*) ((13*b*)), one can see that the linear manifold spanned by the ON subbasis $\{|\phi_i\rangle_n : \forall i\}$ (by $\{|\omega_i\rangle_d : \forall i\}$) in the Schmidt biorthogonal expansion (12) is the *range* $R(\rho_n)$ ($R(\rho_d)$). The subspace spanned by the same subbasis is the (topologically) closed range $\bar{R}(\rho_n)$ ($\bar{R}(\rho_d)$). If the subbasis is infinite, the range is a proper subset of the closed range. Otherwise, they coincide.

Definition 5. Let $|\chi\rangle_{nd}$ be given as a Schmidt biorthogonal expansion (12). Let us denote by $\mathcal{U}_a$ and call the correlation operator (implied by $|\chi\rangle_{nd}$) the antiunitary (antilinear unitary) isomorphism mapping $\bar{R}(\rho_n)$ onto $\bar{R}(\rho_d)$ so that

$$\forall i : \mathcal{U}_a|\phi_i\rangle_n = |\omega_i\rangle_d. \tag{14a}$$

Remark 2. Definition (14*a*) is evidently equivalent to

$$\mathcal{U}_a = \sum_i |\omega_i\rangle_d K \langle\phi_i|_n \tag{14b}$$

where 'K' is the operation of complex conjugation. Thus,

$$\forall|\lambda\rangle_n \ (\in \bar{R}(\rho_n)) \colon \ \mathcal{U}_a|\lambda\rangle_n = \sum_i (\langle\phi_i|_n|\lambda\rangle_n)^* |\omega_i\rangle_d.$$

Lemma 5. Though a Schmidt biorthogonal expansion (12) of a given composite-system state vector $|\chi\rangle_{nd}$ is not unique (the characteristic subbases displayed in it are not unique, though the spectra of the subsystem state operators always are), the correlation operator $\mathcal{U}_a$ is always *uniquely* implied by $|\chi\rangle_{nd}$.

Remark 3. Evidently, one can rewrite (12) in any of the following two forms:

$$|\chi\rangle_{nd} = \sum_i r_i^{1/2} |\phi_i\rangle_n (\mathcal{U}_a|\phi_i\rangle_n)_d \tag{15a}$$

$$|\chi\rangle_{nd} = \sum_i r_i^{1/2} (\mathcal{U}_a^{-1}|\omega_i\rangle_d)_n |\omega_i\rangle_d. \tag{15b}$$

Corollary 9. A given $|\chi\rangle_{nd}$ implies its ρ_n, ρ_d, and $\mathcal{U}_a$. But also vice versa, ρ_n, and $\mathcal{U}_a$, or, alternatively, ρ_d and $\mathcal{U}_a^{-1}$, determine $|\chi\rangle_{nd}$.

Corollary 10. If Q_n and Q_d are the *range projectors* of the respective subsystem state operators ρ_n and ρ_d of a given $|\chi\rangle_{nd}$, then always

$$|\chi\rangle_{nd} = (Q_n \otimes I_d)|\chi\rangle_{nd} = (I_n \otimes Q_d)|\chi\rangle_{nd}.$$

Definition 6. We call the number N of terms in the Schmidt canonical form (12) of a given $|\chi\rangle_{nd}$, or equivalently, the number of dimensions of the (necessarily equally dimensional) ranges $R(\rho_p)$, $p = n, d$, *the degree of entanglement* in $|\chi\rangle_{nd}$. If N is finite, we say that $|\chi\rangle_{nd}$ is finitely entangled; otherwise, we speak of infinite entanglement.

Definition 7. If a given composite-system state vector $|\chi\rangle_{nd}$ is finitely entangled with the degree N, and if $\forall i : r_i = 1/N$ (cf (12)), then one says that the vector is *maximally entangled*.

Definition 8. Let $\boldsymbol{S}_n$ be a subspace of $\mathcal{H}_n$ and let Q_n be the range projector of ρ_n of a given composite-system state vector $|\chi\rangle_{nd}$ (cf (12) and (13*a*)). If $R(Q_n) \subseteq \boldsymbol{S}_n$, then we say that $|\chi\rangle_{nd}$ is *within* $\boldsymbol{S}_n$. If, in particular, $R(Q_n) = \boldsymbol{S}_n$, then $|\chi\rangle_{nd}$ is said to be *complete in* $\boldsymbol{S}_n$. (If $R(Q_n) \subset \boldsymbol{S}_n$, then the vector is incomplete in $\boldsymbol{S}_n$). These definitions are valid symmetrically for the distant subsystem.

Remark 4. Every state vector $|\chi\rangle_{nd}$ is complete in the closed ranges $\bar{R}(\rho_n)$ $(= R(Q_n))$ and $\bar{R}(\rho_d)$ $(= R(Q_d))$, where Q_n and Q_d are the respective range projectors of ρ_n and ρ_d.

In the main result of this section, the square root of the subsystem state operator $\rho_d^{1/2}$ is going to play a decisive role. It is desirable to gain some knowledge on it.

Remark 5. In case of infinite entanglement, the spectral form (13*b*) implies, as easily seen:

$$R(\rho_d) = \{|\lambda\rangle_d : |\lambda\rangle_d \in \mathcal{H}_d, \sum_i (|\omega_i|_d|\lambda\rangle_d|^2/r_i^2) < \infty\} \tag{16}$$

$$R(\rho_d^{1/2}) = \{|\lambda\rangle_d : |\lambda\rangle_d \in \mathcal{H}_d, \sum_i (|\langle\omega_i|_d|\lambda\rangle_d|^2/r_i) < \infty\}. \tag{17}$$

In $\mathcal{H}_n$ one has the symmetric relations.

Remark 6. If $|\chi\rangle_{nd}$ is infinitely entangled, then we have the following chain of ranges as proper subsets:

$$R(\rho_d) \subset R(\rho_d^{1/2}) \subset R(Q_d) \qquad (R(Q_d) = \bar{R}(\rho_d) = \bar{R}(\rho_d^{1/2}))$$

whereas, if the entanglement is finite, then all three linear manifolds coincide. Naturally, in $\mathcal{H}_n$ we have the symmetric relations.

Theorem 2. If a composite-system state vector $|\chi\rangle_{nd}$ is given, then a state vector $|\lambda\rangle_d$ can be obtained by distant preparation (cf proposition 1 and theorem 1) if and only if $|\lambda\rangle_d \in \mathrm{R}(\rho_d^{1/2})$.

Proof. As it is clear from proposition 1 and theorem 1, the (unnormalized) vector $|\lambda\rangle_d'$ is obtainable by distant preparation if and only if there exists a state vector $|\phi\rangle_n$ such that

$$\langle\phi|_n|\chi\rangle_{nd} = |\lambda\rangle_d'. \tag{18}$$

Replacing here (12), one obtains that the characteristic condition is existence of a $|\phi\rangle_n$ that satisfies

$$|\lambda\rangle_d' = \sum_i \langle\phi|_n|\phi_i\rangle_n r_i^{1/2}|\omega_i\rangle_d. \tag{19}$$

To prove the *necessity* of the claimed condition, let us assume that there exists a $|\phi\rangle_n$ satisfying (19). Its very existence makes $(\sum_i |\langle\phi_i|_n|\phi\rangle_n|^2) < \infty$ necessary. Further, (19) entails $\langle\phi|_n|\phi_i\rangle_n = \langle\omega_i|_d|\lambda\rangle_d'^2/r_i^{1/2}$. Substitution in the preceding inequality gives $[\sum_i(|\langle\omega_i|_d|\lambda\rangle_d'|^2/r_i)] < \infty$. On account of (17) this implies

$$|\lambda\rangle_d' \in R(\rho_d^{1/2}). \tag{20}$$

To prove *sufficiency*, we assume the validity of the claimed condition (20). Then, owing to (17), we can define

$$|\phi\rangle_n \equiv \sum_i (\langle\lambda|_d'|\omega_i\rangle_d/r_i^{1/2})|\phi_i\rangle_n$$

and, on account of (19), it determines by distant preparation the given $|\lambda\rangle_d'$. □

This result was derived in previous work (Herbut and Vujičić 1976) using antilinear Hilbert–Schmidt operator techniques. Therefore, the above proof may be viewed as a simplified one.

5. The central theorem on statistical teleportation

As it was stated, in the first step of statistical teleportation one measures $(|\bar{a}\rangle_{12}\langle\bar{a}|_{12} \otimes I_3)$ in the state $|\psi\rangle_1|\bar{\Psi}\rangle_{23}$. The nearby subsystem is $(1+2)$, and the distant one 3. Owing to theorem 2, we know that the distantly prepared state $|\psi\rangle'_3$ (cf the figure) necessarily belongs to the linear manifold $R((\bar{\rho}_3^B)^{1/2})$, where

$$\bar{\rho}_3^B \equiv \mathrm{Tr}_{12}(|\psi\rangle_1|\bar{\Psi}\rangle_{23}\langle\psi|_1\langle\bar{\Psi}|_{23}) = \mathrm{Tr}_2(|\bar{\Psi}\rangle_{23}\langle\bar{\Psi}|_{23}). \tag{21}$$

As it was stated, under the assumption that a bridge state vector $|\bar{\Psi}\rangle_{12}$ is given, we search for a state vector $|\bar{a}\rangle_{12}$ such that it makes possible the first step of statistical teleportation, i.e. distant preparation as a unitary map $\mathcal{U}_{31}$ of $\boldsymbol{S}_1$ onto $\boldsymbol{S}'_3$. The second step is then, in principle, defined as obvious from the figure:

$$\mathcal{U}_3 = \mathcal{I}_{31} \circ \mathcal{U}_{31}^{-1}.$$

We now fix *arbitrarily* $\boldsymbol{S}_1$ as a linear submanifold of $\mathcal{H}_1$ with finite or infinite dimensionality. We introduce the corresponding projector P_1 projecting onto the subspace $\bar{\boldsymbol{S}}_1$ (topological closure of $\boldsymbol{S}_1$).

Further, we define the linear submanifold $\boldsymbol{S}'_3$ of $\mathcal{H}_3$ as the one spanned by all distantly prepared state vectors $|\Psi\rangle'_3$ obtained from some vector $|\psi\rangle_1 \in \boldsymbol{S}_1$, and we denote by P'_3 the projector projecting $\mathcal{H}_3$ onto $\bar{\boldsymbol{S}}'_3$.

Definition 9. Let the *relevant bridge state vector* $|\Psi\rangle_{23}$ be defined by

$$|\Psi\rangle_{23} \equiv c^{-1/2}(I_2 \otimes P'_3)|\bar{\Psi}\rangle_{23} \tag{22}$$

where c is the square norm of the projection. The range projector of the state operator of the second subsystem in the relevant bridge state vector, i.e. of

$$\rho_2^B \equiv \mathrm{Tr}_3\, |\Psi\rangle_{23}\langle\Psi|_{23} \tag{23}$$

will be denoted by P_2.

To make sure that definition (22) is consistent, we must ascertain of the following fact.

Lemma 6. The projection $(I_2 \otimes P'_3)|\bar{\Psi}\rangle_{23}$ is never zero.

Proof. Assuming *ab contrario* that the projection is zero, and utilizing (21), we obtain

$$0 = \mathrm{Tr}_2(I_2 \otimes P'_3)|\bar{\Psi}\rangle_{23}\langle\bar{\Psi}|_{23}(I_2 \otimes P'_3) = P'_3\bar{\rho}_3^B P'_3.$$

(We have 'pulled out' P'_3 from Tr_2 (cf corollary 6)). Theorem 2 and the definition of $\boldsymbol{S}'_3$ imply $R(P'_3) \subseteq R[(\bar{\rho}_3^B)^{1/2}]$. Hence, if $P'_3|\phi\rangle_3 = |\phi\rangle_3$, then, on the one hand, $[(\bar{\rho}_3^B)^{1/2}]$ acts on $|\phi\rangle_3$ as a nonsingular operator, and, on the other hand, the above relation implies

$$0 = \langle\phi|_3(P'_3\bar{\rho}_3^B P'_3)|\phi\rangle_3 = \|((\bar{\rho}_3^B)^{1/2}|\phi\rangle_3\|^2$$

and, further, $(\bar{\rho}_3^B)^{1/2}|\phi\rangle_3 = 0$ in contradiction with the above. This completes our *reductio ad absurdum* proof. □

Definition 10. We call the state vector $|a\rangle_{12}$ defined by

$$|a>_{12} \equiv d^{-1/2}(P_1 \otimes P_2)|\bar{a}\rangle_{12} \tag{24}$$

where d is the square norm of the projection, *the relevant atom state vector*.

Lemma 7. The basic distant-preparation expression can be reduced to an analogous expression in terms of the relevant state vectors:

$$\forall|\psi\rangle_1 \in \boldsymbol{S}_1 : \langle\bar{a}|_{12}(|\psi\rangle_1|\bar{\Psi}\rangle_{23}) = c^{1/2}d^{1/2}\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}). \tag{25}$$

Proof. Utilizing first the fact that, by definition of $\boldsymbol{S}_3'$, all distantly prepared vectors belong to it, then corollary 4, then (22), then $|\psi\rangle_1 \in \boldsymbol{S}_1$ and the definition of P_2 together with corollary 10, then corollary 3, and finally (24), one can write:

$$\begin{aligned}\text{LHS} &= P_3' \langle \bar{a}|_{12}(|\psi\rangle_1|\bar{\Psi}\rangle_{23}) = \langle \bar{a}|_{12}\{|\psi\rangle_1[(I_2 \otimes P_3')|\bar{\Psi}\rangle_{23})]\} \\ &= c^{1/2}\langle \bar{a}|_{12}(|\psi\rangle_1|\Psi\rangle_{23}) = c^{1/2}\langle \bar{a}|_{12}[(P_1|\psi\rangle_1)(P_2 \otimes I_3)|\Psi\rangle_{23}] \\ &= c^{1/2}[\langle \bar{a}|_{12}(P_1 \otimes P_2)](|\psi\rangle_1|\Psi\rangle_{23}) = \text{RHS}.\end{aligned}$$

□

Proposition 2. If the measurement of the event $(|\bar{a}\rangle_{12}\langle \bar{a}|_{12} \otimes I_3)$ in the state $|\psi\rangle_1|\bar{\Psi}\rangle_{23}$ gives (by distant preparation) the state vector $|\psi\rangle_3'$ with probability $\bar{w}$, then the same state vector $|\psi\rangle_3'$ can be obtained with a probability w measuring the event $(|a\rangle_{12}\langle a|_{12} \otimes I_3)$ in the state $|\psi\rangle_1|\Psi\rangle_{23}$, where $|a\rangle_{12}$ is the relevant atom state vector of $|\bar{a}\rangle_{12}$ (cf (24)), and $|\Psi\rangle_{23}$ is the relevant bridge state vector (cf (22)). Besides, one has the relation:

$$\bar{w} = wcd. \tag{26}$$

Proof. Let us use the basic distant preparation formula (4) as the first step of teleportation in the two versions at issue:

$$\langle \bar{a}|_{12}(|\psi\rangle_1|\bar{\Psi}\rangle_{23}) \equiv (\bar{w})^{1/2}|\bar{\psi}\rangle_3' \tag{27a}$$

$$\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}) = w^{1/2}|\psi\rangle_3'. \tag{27b}$$

As an immediate consequence of lemma 7, one can write

$$|\bar{\psi}\rangle_3' = (\bar{w})^{-1/2}c^{1/2}d^{1/2}\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}) = (\bar{w})^{-1/2}c^{1/2}d^{1/2}w^{1/2}|\psi\rangle_3'.$$

Since we are dealing with state vectors and positive numbers, the claims $|\bar{\psi}\rangle_3' = |\psi\rangle_3'$ and (26) follows. □

Remark 7. The distant-preparation relation (27*a*) is connected with the actual measurement (in the laboratory) of the event $(|\bar{a}\rangle_{12}\langle \bar{a}|_{12} \otimes I_3)$ in the state $|\psi\rangle_1|\bar{\Psi}\rangle_{23}$, i.e. with the first step of *actual* teleportation. Relation (27*b*), on the other hand, corresponds to *virtual* measurement of $(|a\rangle_{12}\langle a|_{12} \otimes I_3)$ in the state $|\psi\rangle_1|\Psi\rangle_{23}$. We call it 'virtual' because it is not actually performed, but it is conceivable.

We proceed in our search for atomic state vectors $|\bar{a}\rangle_{12}$ that will give teleportation. Further on, we utilize (27*b*) and virtual measurement, and we concentrate on its interpretation as a linear map $\mathcal{U}_{31}$.

Lemma 8. The relevant distant preparation relation (27*b*) amounts to a linear map $\mathcal{U}_{31}$ taking $\boldsymbol{S}_1$ into $\boldsymbol{S}_3'$ (cf (6*b*)) *if and only if* the probability w is one and the same positive number for all state vectors $|\psi\rangle_1 \in \boldsymbol{S}_1$.

Proof. For $|\psi\rangle_1, |\phi\rangle_1, |\omega\rangle_1 \in \boldsymbol{S}_1$, $\langle\psi|_1|\psi\rangle_1 = \langle\phi|_1|\phi\rangle_1 = \langle\omega|_1|\omega\rangle_1 = 1$, and $|\omega\rangle_1 = \alpha|\psi\rangle_1 + \beta|\phi\rangle_1, \alpha, \beta \in \boldsymbol{C}$, one can write

$$\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}) = w^{1/2}|\psi\rangle_3' \tag{28a}$$

$$\langle a|_{12}(|\phi\rangle_1|\Psi\rangle_{23}) = x^{1/2}|\phi\rangle_3' \qquad \langle a|(|\omega\rangle|\Psi\rangle) = y^{1/2}|\omega\rangle' \tag{28b}$$

and

$$\alpha\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}) + \beta\langle a|_{12}(|\phi\rangle_1|\Psi\rangle_{23}) = y^{1/2}|\omega\rangle_3'. \tag{29}$$

Sufficiency. We assume $w^{1/2} = x^{1/2} = y^{1/2} > 0$. Substituting on the LHS of (29) the equalities (28a), (28b), after cancellation one has

$$|\omega\rangle_3' = \alpha|\psi\rangle_3' + \beta|\phi\rangle_3' \tag{30}$$

as claimed.

Necessity. Let (30) be valid by assumption. Replacing (28a), (28b) and (30) in (29) one obtains

$$\alpha w^{1/2}|\psi\rangle_3' + \beta x^{1/2}|\phi\rangle_3' = \alpha y^{1/2}|\psi\rangle_3' + \beta y^{1/2}|\phi\rangle_3'.$$

If $|\psi\rangle_3'$ and $|\phi\rangle_3'$ are linearly independent, then the uniqueness of the expansion coefficients, and the specification $\alpha \neq 0 \neq \beta$ imply $w = x = y$. If we take $\alpha \neq 0 = \beta$ (or $\alpha = 0 \neq \beta$), then $|\omega\rangle_3'$, and $|\psi\rangle_3'$ (or $|\phi\rangle_3'$) are collinear, and it is clear from the distant-preparation relation (say (28a)) that the probability is the same. The positivity $w > 0$ is necessary, because otherwise $\mathcal{U}_{31}$ would map $\boldsymbol{S}_1$ into zero. $\square$

We still do not know what kind of atomic state vector $|\bar{a}\rangle_{12}$ will give constant w over $\boldsymbol{S}_1$ in the relevant distant-preparation relation (27b). For further investigation, the Schmidt biorthogonal expansion (cf (12)) of the relevant bridge state vector $|\Psi\rangle_{23}$ is desirable. Let it be

$$|\Psi\rangle_{23} = \sum_i r_i^{1/2}|i\rangle_2|i\rangle_3. \tag{31a}$$

It is accompanied by the spectral forms of the subsystem state operators

$$\rho_2^B = \sum_i r_i|i\rangle_2\langle i|_2 \tag{31b}$$

$$\rho_3^B \equiv \mathrm{Tr}_2\,|\Psi\rangle_{23}\langle\Psi|_{23} = \sum_i r_i|i\rangle_3\langle i|_3 \tag{31c}$$

$\forall i : r_i > 0$ (cf (23) and (13a), (13b) *mutatis mutandis*).

Lemma 9. The relevant distant preparation relation (27b), i.e.

$$\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}) = w^{1/2}|\psi\rangle_3' \tag{32}$$

amounts to a unitary map $|\psi\rangle_3' = \mathcal{U}_{31}|\psi\rangle_1$ *if and only if* the following *suitability relation* is satisfied:

$$\rho_2 \equiv \mathrm{Tr}_1\,|a\rangle_{12}\langle a|_{12} = w(\tilde{\rho}_2^B)^{-1}P_2 \tag{33}$$

where $\tilde{\rho}_2^B$ is the restriction of ρ_2^B to its range $R(P_2)$. This implies

$$w = \{\mathrm{Tr}[(\tilde{\rho}_2^B)^{-1}]\}^{-1}. \tag{34}$$

Proof. It is useful to expand $|a\rangle_{12}$ in the ON subbasis $\{|i\rangle_2 : \forall i\}$ spanning $\boldsymbol{S}_2$ that is a characteristic basis of $\tilde{\rho}_2^B$:

$$|a\rangle_{12} = \sum_i (r_i^a)^{1/2}|i\rangle_1|i\rangle_2. \tag{35}$$

The subbasis $\{|i\rangle_1 : \forall i\}$ in $\boldsymbol{S}_1$ ($|a\rangle_{12} \in \boldsymbol{S}_1 \otimes \boldsymbol{S}_2$, cf (24)) *a priori* need not be orthogonal. We require the vectors $|i\rangle_1$ to be normalized, i.e. $\forall i : (r_i^a)^{1/2}$ is, by definition, the norm of the generalized expansion coefficient $(r_i^a)^{1/2}|i\rangle_1$ in (35) (cf lemma 2 *mutatis mutandis*).

Replacing expansions (35) and 31a) in the relevant distant-preparation relation (32), one obtains:

$$\sum_i \sum_{i'} [(r_i^a)^{1/2} r_{i'}^{1/2} \langle i|_1|\psi\rangle_1 \langle i|_2|i'\rangle_2]|i'\rangle_3 = w^{1/2}|\psi\rangle_3'.$$

Since $\langle i|_2|i'\rangle_2 = \delta_{i,i}$, one ends up with

$$\sum_i (r_i^a)^{1/2} r_i^{1/2} \langle i|_1|\psi\rangle_1 |i\rangle_3 = w^{1/2}|\psi\rangle_3'$$

or equivalently,

$$\forall i : w^{-1/2}(r_1^a)^{1/2} r_i^{1/2} \langle i|_1|\psi\rangle_1 = \langle i|_3|\psi\rangle_3'. \tag{36}$$

Since all steps that led from (32)–(36), made with the help of the expansions (31a) and (35), can be performed backwards, (36) is an equivalent form of (32). We must clarify under which conditions can (36) be viewed as a unitary isomorphism $\mathcal{U}_{31}$ mapping $\boldsymbol{S}_1$ onto $\boldsymbol{S}_3'$.

To prove *necessity* of the claimed condition (33), we assume that (36) determines a unitary isomorphism $\mathcal{U}_{31}$ ($\mathcal{U}_{31}|\psi\rangle_1 = |\psi\rangle_3'$) mapping $\boldsymbol{S}_1$ onto $\boldsymbol{S}_3'$.

Making $|\psi\rangle_1$ run over the ON basis vectors $\{|\bar{i}\rangle_1 \equiv \mathcal{U}_{31}^{-1}|i\rangle_3 : \forall i\}$, and taking into account the fact that w is one and the same by this (cf lemma 8), (36) implies

$$w^{-1/2}(r_i^a)^{1/2} r_i^{1/2} \langle i|_1 \bar{i}'\rangle_1 = \delta_{i,i'}.$$

This makes the vectors $\{|i\rangle_1 : \forall i\}$, which were assumed to be normalized, also orthogonal on account of the orthonormality of the basis $\{|\bar{i}\rangle_1 : \forall i\}$, and it makes (35) a biorthogonal expansion. Besides, the last relation also implies

$$\forall i : r_i^a = w r_i^{-1}.$$

Since all r_i^a are seen to be positive, (35) is a Schmidt biorthogonal expansion (cf definition 4). Then, in view of the spectral form

$$\rho_2 = \sum_i r_i^a |i\rangle_1 \langle i|_1$$

which we can now read from (35) (cf corollary 8 *mutatis mutandis*), and taking into account (31b), the claimed relation (33) follows.

To prove *sufficiency* of the claimed condition (33), we point out that, due to the stated equivalence of (36) and (32), w in the two relations is one and the same, and hence it is the probability of distant preparation. Now, (33) implies (34), and hence the independence of w from the choice of $|\psi\rangle_1$ in $\boldsymbol{S}_1$. According to lemma 8, this is sufficient for the linearity (and hence the unitarity) of $\mathcal{U}_{31}$ determined by (32). □

Now we are prepared to state the central result of this study.

Theorem 3. Let $|\bar{\Psi}\rangle_{23}$ be a given (finitely or infinitely) entangled bridge state vector. Let the linear manifold $\boldsymbol{S}_1$ ($\subseteq \mathcal{H}_1$) be arbitrarily chosen but finitely dimensional, its number of dimensions not exceeding that of $R(\bar{\rho}_3^B)$. Then and only then there exist atomic state vectors $|\bar{a}\rangle_{12}$ such that the first step of statistical teleportation is possible.

A necessary and sufficient condition for the mentioned atomic state vector is that it defines a relevant atom state vector (i.e. that $d > 0$ in (24)), and that the latter, if written as a Schmidt biorthogonal expansion (35), satisfies:

(i) $\{|i\rangle_2 : \forall i\}$ is an arbitrary characteristic ON basis of ρ_2^B in $\boldsymbol{S}_2$;

(ii) $\forall i : r_i^a = w r_i^{-1}$ (cf (31a)–(31c)), and

(iii) w, the probability of statistical teleportation, is given by (34), i.e. by

$$w = \{\mathrm{Tr}[(\tilde{\rho}_2^B)^{-1}]\}^{-1}.$$

Proof. Proof follows immediately from lemma 9, because, in view of the spectral forms of ρ_2^B and ρ_2, (i) in conjunction with (ii) is equivalent to (33). The subspace $\boldsymbol{S}_1$ has to be finitely dimensional, because, in the first step of teleportation, $\boldsymbol{S}_1$, $\boldsymbol{S}_3'$ and $\boldsymbol{S}_2$ were seen to be equally dimensional (cf (6*b*) and (31*a*)), and for infinitely dimensional $\boldsymbol{S}_2$ the inversion in (33) would not be possible. Since $\boldsymbol{S}_3' \subseteq R(\bar{\rho}_3^B)$ (cf theorem 2 and remark 6), the number of dimensions of the former must not exceed that of the latter. These properties are seen to be also sufficient for (33), which, according to lemma 9, is sufficient for $|\bar{a}\rangle_{12}$ to be atomic if $d > 0$ in (24). □

Remark 8. Note that, while the spectrum $\{r_j^a : \forall j\}$ and the ON subbasis $\{|i\rangle_2 : \forall i\}$ in the Schmidt biorthogonal expansion (35) are determined by ρ_2^B, the ON subbasis $\{|i\rangle_1 : \forall i\}$ is an (arbitrary) ON basis in $\boldsymbol{S}_1$. Also the irrelevant parts of the atomic state vector $|\bar{a}\rangle_{12}$, i.e.

$$(P_1 \otimes P_2^{\perp})|\bar{a}\rangle_{12} \qquad (P_1^{\perp} \otimes P_2)|\bar{a} >_{12} \qquad \text{and} \qquad (P_1^{\perp} \otimes P_2^{\perp})|\bar{a}\rangle_{12}$$

($P_2^{\perp}$ being the orthocomplementary projector of P_2 in $\mathcal{H}_2$ etc) are completely arbitrary (cf (24)). This may prove useful in practice (which will hopefully be reached): it may help to find tractable atomic state vectors $|\bar{a}\rangle_{12}$.

Finally, the unitary isomorphism $\mathcal{U}_3$ (taking the subspace $\boldsymbol{S}_3'$ onto $\boldsymbol{S}_3$ ($\equiv \mathcal{I}_{31}\boldsymbol{S}_1$)), responsible for the second step of teleportation, is implied (as it was stated) as $\mathcal{U}_3 = \mathcal{I}_{31} \circ \mathcal{U}_{31}^{-1}$.

One should note that the larger the subspace $\boldsymbol{S}_1$ is, the more statistical teleportation can be accomplished with one and the same $|\bar{a}\rangle_{12}$ and one and the same $\mathcal{U}_3$. Also the probability $\bar{w}$ is one and the same.

Finally, it is important to point out that the central result is immediately applicable to deterministic teleportation. Namely, the probability w does not depend on the choice of the atom state. (It is implied solely by the bridge state.) Hence, it is valid simultaneously for all, what may be called, characteristic atomic state vectors of a deterministically teleporting observable (cf (9)).

The operator relation (33) is thus the searched for *necessary and sufficient condition* also for the observable in deterministic teleportation.

6. Immediate consequences of the central theorem

Corollary 11. If one denotes by $\mathcal{U}_a^B$ and $\mathcal{U}_a$ the correlation operator of the relevant bridge state vector $|\Psi\rangle_{23}$ and that of the relevant atom state vector $|a\rangle_{12}$ respectively (cf definition 5), then one has

$$\mathcal{U}_a = (\mathcal{U}_a^B)^{-1} \circ \mathcal{U}_{31} \tag{37}$$

(see the commuting lower triangle on figure 2 below).

Proof. The suitability relation (33) of the atomic state vector implies that each characteristic basis of ρ_2^B is also a characteristic basis of ρ_2. Let $\{|i\rangle_2 : \forall i\}$ be a common characteristic

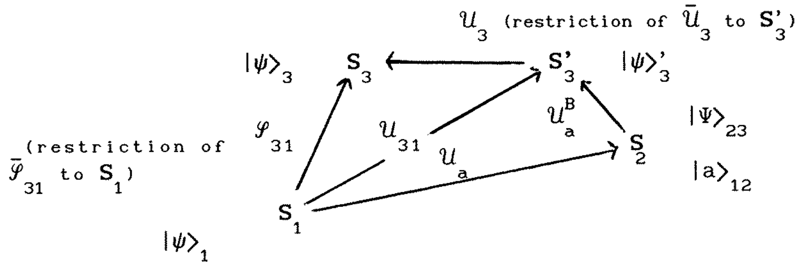

Figure 2.

orthonormal subbasis of ρ_2^B and ρ_2 spanning their common range $\boldsymbol{S}_2$. Then, utilizing (33), we can write down the Schmidt biorthogonal expansions as follows

$$|\Psi\rangle_{23} = \sum_{i=1}^{N} r_i^{1/2} |i\rangle_2 \otimes [(\mathcal{U}_a^B)|i\rangle_2]_3 \tag{38}$$

$$|a\rangle_{12} = \sum_{i=1}^{N} w^{1/2} r_i^{-1/2} [(\mathcal{U}_a)^{-1}|i\rangle_2]_1 \otimes |i\rangle_2 \tag{39}$$

(cf (15*a*) and (15*b*) *mutatis mutandis*).

By the very definition of $\mathcal{U}_{31}$ (and on account of proposition 2, stating that distant preparation can be written in terms of the relevant state vectors), one has:

$$\mathcal{U}_{31}|\psi\rangle_1 = |\psi\rangle'_3 = w^{-1/2}\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}).$$

Substitution of $|\psi\rangle_1 \equiv [(\mathcal{U}_a^{-1})|i\rangle_2]_1$, of (38), and of (39) entails

$$\forall i : \mathcal{U}_{31}[(\mathcal{U}_a^{-1})|i\rangle_2]_1 = [(\mathcal{U}_a^B)|i\rangle_2]_3.$$

On account of the commutativity of the triangle of unitary maps at issue (see figure 2), the claimed relation (37) then clearly follows. □

Corollary 12. In view of (34) and (38), one can write

$$w = 1 / \sum_{i=1}^{N} r_i^{-1} \tag{40}$$

where N is the degree of entanglement of the relevant bridge state vector (cf definition 6 and (38)). It is always true that

$$w \leqslant 1/N^2 \tag{41}$$

and the maximal value $w = 1/N^2$ is reached if and only if the relevant bridge state vector is *maximally entangled*, i.e. $\forall i : r_i = 1/N$ (cf definition 7).

Proof. Relation (40) is the explicit form of (34). It is the harmonic mean value of the characteristic values r_i (of $\tilde{\rho}_2^B$) divided by N. Since the maximal value of the harmonic mean is the arithmetic mean, one obtains (41). The last claim follows from the fact that the maximal value of the harmonic mean, at fixed value of the arithmetic mean, is achieved if and only if the characteristic values are equal to each other. Since $\sum_i r_i = 1$, we are then dealing with a maximally entangled bridge state vector as claimed. □

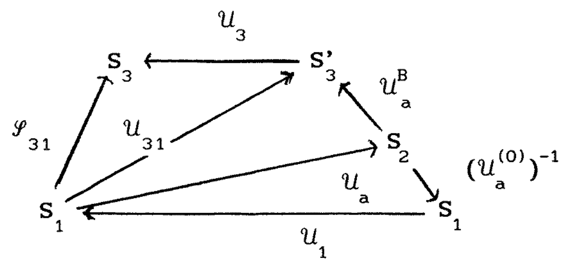

Figure 3.

Thus, from the statistical point of view it is most favourable to use a maximally entangled relevant bridge state vector for statistical teleportation because it gives the largest probability of teleportation.

Corollary 13. Let $|\bar{a}\rangle_{12}$ be an atomic state vector in $(\mathcal{H}_1 \otimes \mathcal{H}_2)$, i.e. a state vector such that (24) with $d > 0$ and (33) are valid. Let, further, $\mathcal{U}_a$ be the correlation operator of the implied atom state vector $|a\rangle_{12}$ mapping the range $R(\rho_1)$, where $\rho_1 \equiv \mathrm{Tr}_2\, |a\rangle_{12}\langle a|_{12}$, onto $R(\rho_2)$ (cf (39)). Then, *the relevant part* $\mathcal{U}_3$ of the operation $\bar{\mathcal{U}}_3$, i.e. the map obtained by restricting the domain of the latter to $\boldsymbol{S}'_3$ $(\equiv R(\rho_3^B))$, is a unitary isomorphism mapping $\boldsymbol{S}'_3$ onto $\boldsymbol{S}_3$ $(\equiv \mathcal{I}_{31} R(\rho_1))$. The isomorphism $\mathcal{U}_3$ is uniquely determined by the two commuting triangles on figure 2 as follows

$$\mathcal{U}_3 = \mathcal{I}_{31} \circ \mathcal{U}_{31}^{-1} \qquad \mathcal{U}_{31} = \mathcal{U}_a^B \circ \mathcal{U}_a. \tag{42}$$

Proof. Obvious in view of the facts that the central theorem covers all possible atomic state vectors $|\bar{a}\rangle_{12}$ for fixed $\boldsymbol{S}_1$, and that the two basic ingredient entities ρ_2 and $\mathcal{U}_a$ of the relevant atom state vector $|a\rangle_{12}$ are decoupled (cf corollary 9). Note that figure 2, on which the mutual connections between the maps are displayed, is completely independent of ρ_2^B, which determines ρ_2. □

Corollary 14. Let $|\bar{a}^{(0)}\rangle_{12}$ be a fixed atomic state vector, and let $\mathcal{U}_a^{(0)}$ be the correlation operator of the implied relevant atom state vector $|a^{(0)}\rangle_{12}$ (cf definition 5). An arbitrary unitary operator $\mathcal{U}_1$ in $\boldsymbol{S}_1$ gives another relevant atom state vector $|a\rangle_{12}$ by specifying its correlation operator $\mathcal{U}_a$ as follows

$$\mathcal{U}_a^{-1} \equiv \mathcal{U}_1 \circ (\mathcal{U}_a^{(0)})^{-1} \tag{43}$$

(see figure 3) and ρ_2 is the same both for $|a^{(0)}\rangle_{12}$ and for $|a\rangle_{12}$, cf (33)). (The irrelevant parts of $|\bar{a}\rangle_{12}$, are, of course, arbitrary.) Every $|a\rangle_{12}$ can be obtained in this way.

Finally, the atom-generating unitary operators $\mathcal{U}_1$ and the operations $\mathcal{U}_3$ determine each other along the peripheral route on the commuting figure 3:

$$\mathcal{U}_3 = \mathcal{I}_{31} \circ \mathcal{U}_1 \circ (\mathcal{U}_a^{(0)})^{-1} \circ (\mathcal{U}_a^B)^{-1} \tag{44a}$$

$$\mathcal{U}_1 = \mathcal{I}_{31}^{-1} \circ \mathcal{U}_3 \circ \mathcal{U}_a^B \circ \mathcal{U}_a^{(0)}. \tag{44b}$$

Proof. Obvious. □

Corollary 15. If $|a^{(0)}\rangle_{12}$ is a given relevant atom state vector, and $\mathcal{U}_1$ is an arbitrary unitary operator in $\boldsymbol{S}_1$, then $|a\rangle_{12} \equiv (\mathcal{U}_1 \otimes I_2)|a^{(0)}\rangle_{12}$ is another relevant atom state vector, actually, the one described in the preceding corollary.

Proof. Follows immediately from (43) when one writes one relevant atom state vector in its Schmidt biorthogonal expansion, and one derives that of the other:

$$|a\rangle_{12} = \sum_i w^{1/2} r_i^{-1/2} (\mathcal{U}_a^{-1}|i\rangle_2)_1 |i\rangle_2 = \sum_i w^{1/2} r_i^{-1/2} \{(\mathcal{U}_1 \circ (\mathcal{U}_a^{(0)})^{-1})|i\rangle_2\}_1 |i\rangle_2$$
$$= (\mathcal{U}_1 \otimes I_2)|a^{(0)}\rangle_{12}.$$

□

7. Deterministic teleportation

We now return to the important concept of deterministic teleportation invented by Bennet *et al* (1993) (introduced in section 3). A complete observable A_{12} for subsystem $(1+2)$ is given such that all its characteristic vectors are atomic state vectors (see relation (9)):

$$A_{12} = \sum_{m=1}^{N} a_m |\bar{a}^{(m)}\rangle_{12} \langle \bar{a}^{(m)}|_{12} \tag{45a}$$

with $m \neq m' \Rightarrow a_m \neq a_{m'}$, and

$$\langle \bar{a}^{(m)}|_{12} |\bar{a}^{(m')}\rangle_{12} = \delta_{m,m'}. \tag{45b}$$

Since A_{12} is a complete observable in $(\mathcal{H}_1 \otimes \mathcal{H}_2)$ ($\mathcal{H}_i$ being the state space of subsystem i, $i = 1, 2$), the set $\{|\bar{a}^{(m)}\rangle_{12} : m = 1, 2, \ldots, M\}$ must span this space. Hence, M must equal the product of the numbers of dimensions of $\mathcal{H}_1$ and of $\mathcal{H}_2$ (finite or infinite).

Theorem 4. Deterministic teleportation with an entangled bridge state vector $|\bar{\Psi}\rangle_{23}$ and with a finitely-dimensional subspace $\boldsymbol{S}_1$ is possible *only if* the following condition is satisfied:

$$\sum_{m=1}^{M} d_m = w^{-1} c^{-1} \tag{46}$$

where w is given by (34), c is given by (22), and d_m is determined by

$$d_m^{1/2} |a^{(m)}\rangle_{12} \equiv (P_1 \otimes P_2)|\bar{a}^{(m)}\rangle_{12} \qquad m = 1, 2, \ldots, M \tag{47}$$

(cf (24)).

Proof. One has deterministic teleportation if and only if the observable A_{12} given in spectral form by (45a) is complete and each of its characteristic vectors is an atomic state vector. Then one of the atomic events $(|\bar{a}^{(m)}\rangle_{12}\langle \bar{a}^{(m)}|_{12} \otimes I_3)$ necessarily does occur in the measurement of the observable in the state $|\psi\rangle_1 |\bar{\Psi}\rangle_{23}$. Hence, the corresponding probabilities $\bar{w}_m$ $(= wcd_m$, cf (26)) of the mentioned events must add up into 1. This can obviously be written as (46). □

Remark 9. If there is no redundancy in the characteristic atomic state vectors $|\bar{a}^{(m)}\rangle_{12}$ of a deterministically teleporting observable A_{12} (cf (45a), (45b)), i.e. if $|\bar{a}^{(m)}\rangle_{12} = |a^{(m)}\rangle_{12}$, $m = 1, 2, \ldots, M$ (cf (24)), then, on the one hand, each of the vectors belongs to $(\boldsymbol{S}_1 \otimes \boldsymbol{S}_2)$, and on the other hand, A_{12} must be complete in $(\mathcal{H}_1 \otimes \mathcal{H}_2)$, or, equivalently the basis $\{|a^{(m)}\rangle_{12} : m = 1, 2, \ldots, M\}$ must span the entirety of this space. Hence, necessarily $\boldsymbol{S}_1 = \mathcal{H}_1$, and $\boldsymbol{S}_2 = \mathcal{H}_2$, and $\mathcal{H}_k$, $k = 1, 2$, must be *finite dimensional*. They must also be *equally dimensional* because the number of dimensions of both must equal that of $\boldsymbol{S}_3'$ (on account of $\mathcal{U}_{31}$ and the definition of $\boldsymbol{S}_2$).

If there is redundancy, then, it appears, there is the possibility of using any entangled bridge state vector $|\bar{\Psi}\rangle_{23}$, even an infinitely entangled one, for deterministic teleportation, as well as an infinitely dimensional state space $\mathcal{H}_1$. Namely, however large the RHS of (46), the numerous small d_m might add up into it.

Theorem 5. In the special case when $\boldsymbol{S}_1 = \mathcal{H}_1$ and $\boldsymbol{S}_2 = \mathcal{H}_2$, deterministic teleportation is possible if and only if there is no redundancy either in the bridge state vector or in any characteristic state vector, and if the former is *maximally entangled*.

Proof. Let N be the degree of entanglement of $|\Psi\rangle_{23}$ (and, of course, the number of dimensions of $\boldsymbol{S}_i, i = 1, 2$). Relation (46) reads $\sum_{m=1}^{M} w d_m c = 1$, and $M = N^2$. Since

$$w \leqslant 1/N^2 \qquad c \leqslant 1 \qquad d_m \leqslant 1 \qquad m = 1, 2, \ldots, N^2$$

(cf (41), (22), and (47)), relation (46) can be satisfied if and only if each term has its maximal value $(1/N^2)$, i.e.

$$w = 1/N^2 \qquad c = d_m = 1 \qquad m = 1, 2, \ldots, N^2$$

and the first of these conditions means maximal entanglement (cf corollary 12). □

Corollary 16. Let again $\boldsymbol{S}_1 = \mathcal{H}_1, \boldsymbol{S}_2 = \mathcal{H}_2$. Let, further, $\{|i\rangle_2 : i = 1, 2, \ldots, N\}$ be a characteristic orthonormal basis of $\tilde{\rho}_2^B$ $(= \rho_2^B$, cf (23)). It is a characteristic orthonormal basis also of ρ_2 of each $|\bar{a}^{(m)}\rangle_{12}$ on account of (33). Hence, expansion of each $|\bar{a}^{(m)}\rangle_{12}$ in this basis is a Schmidt biorthogonal expansion (cf lemma 4 and definition 4):

$$|\bar{a}^{(m)}\rangle_{12} = |a^{(m)}\rangle_{12} = N^{-1/2} \sum_{i=1}^{N} |i^{(m)}\rangle_1 |i\rangle_2 \qquad m = 1, 2, \ldots, N^2 \tag{48}$$

(cf theorem 5). The *orthogonality* (45*b*) is equivalent to the $N(N-1)/2$ conditions

$$\forall (m < m') : \qquad \sum_{i=1}^{N} \langle i^{(m)}|_1 |i^{(m')}\rangle_1 = 0. \tag{49}$$

Proof. Immediately obtained when one replaces (48) in (45*b*). □

Corollary 17. If one has one observable A_{12} suitable for deterministic teleportation, any unitary operator $\mathcal{U}_1$ and $\mathcal{H}_1$ generates another:

$$A'_{12} \equiv (\mathcal{U}_1 \otimes I_2) A_{12} (\mathcal{U}_1^\dagger \otimes I_2).$$

Proof. This is an immediate consequence of (48) and (49). □

Corollary 18. If, under the conditions of theorem 5, one fixes one of the characteristic atom state vectors, e.g. $|a^{(m_0)}\rangle_{12}$, and one wants to generate the rest of the $(N^2 - 1)$ characteristic atom state vectors of a deterministically teleporting observable A_{12} out of it using unitary operators $\mathcal{U}_1^{(m)}$ (cf corollary 5), then *the orthogonality relations* (49) take on the *equivalent* form

$$\mathrm{Tr}(\mathcal{U}_1^{(m)})^\dagger \mathcal{U}_1^{(m')} = 0 \qquad m \neq m' \qquad m, m' = 1, 2, \ldots, N^2. \tag{50}$$

Since $\mathcal{U}_1^{(m_0)} = I_1$, (50) is seen to imply

$$\mathrm{Tr}\,\mathcal{U}_1^{(m)} = 0 \qquad \forall m, m \neq m_0. \tag{51}$$

Proof. Follows immediately from (49). □

Evidently, in our search for suitable pairs

$$\{(|a^{(m)}\rangle_{12}, \mathcal{U}_3^{(m)}), m = 1, 2, \ldots, N^2\}$$

for deterministic teleportation one can choose the unitary operators $\mathcal{U}_3^{(m)}$ as the *starting entities* and evaluate $(\mathcal{U}_a^{(m)})$ from (42). Then, under the conditions of theorem 5, the orthogonality relations (49) can be further transformed. Since, moving on the periphery of figure 3, one can write

$$\mathcal{U}_3^{(m)} = \mathcal{I}_{31} \circ \mathcal{U}_1^{(m)} \circ (\mathcal{U}_a^{(m_0)})^{-1} \circ (\mathcal{U}_a^B)^{-1} \qquad m = 1, 2, \ldots, N^2 \tag{52}$$

and since all the factors are unitary or antiunitary isomorphisms, which implies that adjoining is the same as inversion, one obtains

$$\begin{gathered}[(\mathcal{U}_3^{(m)})^\dagger \mathcal{U}_3^{(m')}] = (\mathcal{U}_a^B \circ \mathcal{U}_a^{(m_0)}) \circ [(\mathcal{U}_1^{(m)})^\dagger \mathcal{U}_1^{(m')}] \circ (\mathcal{U}_a^B \circ \mathcal{U}_a^{(m_0)})^{-1} \\ m \neq m' \qquad m, m' = 1, 2, \ldots, N^2.\end{gathered} \tag{53}$$

Thus, the square-bracketed operators are unitary transforms of each other, hence, their traces are equal. This proves the following result.

Corollary 19. Under the conditions of theorem 5, the orthogonality relations (45*b*) are equivalent to

$$\mathrm{Tr}[(\mathcal{U}_3^{(m)})^\dagger \mathcal{U}_3^{(m')}] = 0 \qquad m \neq m' \qquad m, m' = 1, 2, \ldots, N^2. \tag{54}$$

Remark 10. If one solves the orthogonality relations (54), then one derives the ON basis $\{|i^{(m)}\rangle_1 : \forall i\}$ that gives $|a^{(m)}\rangle_{12}$ via (48) utilizing (52) solved for $\mathcal{U}_1^{(m)}$:

$$\forall m, \forall i: \qquad |i^{(m)}\rangle_1 \equiv [(\mathcal{I}_{31}^{-1}) \circ (\mathcal{U}_3^{(m)}) \circ (\mathcal{U}_a^B) \circ (\mathcal{U}_a^{(m_0)})]|\mathrm{i}^{(m_0)}\circ\rangle_1. \tag{55}$$

Remark 11. Let us, for a moment, consider statistical teleportation by itself (not as a constituent of deterministic teleportation). Then, one must realize that if the event determined by the atomic state vector does not occur in its measurement, the state of subsystem 1 changes, and the same individual sample of subsystem 1 is not available for a second attempt. If the measurement is ideal (cf Messiah 1961, Lüders 1951), one can work out this change of state. (But it is not worth doing.) If the measurement is not ideal (as most realistic cases are), then even in principle one does not know the change of state.

Remark 12. It is important to bear in mind that for distant preparation (and for teleportation obtained after the operation $\mathcal{U}_3$ is applied) the measurement of the deterministically teleporting observable *need not be an ideal one* (cf section 6(B) in Herbut and Vujičić 1976).

Remark 13. As an example of derivation of deterministically teleporting observables A_{12} under the conditions of theorem 5, we take $N = 2$, and give a family of ON bases $\{\{|i^{(m)}\rangle_1 : i = 1, 2\} : m = 1, 2, 3, 4\}$ (which then, using (48) and satisfying (45*b*) via (49), each define an observable (45*a*), where the corresponding operations $\mathcal{U}_3^{(m)}$ are given by (52)):

$$\{|{\uparrow}\rangle_1, |{\downarrow}\rangle_1\} \qquad \{\mathrm{e}^{\mathrm{i}\theta}|{\downarrow}\rangle_1, \mathrm{e}^{-\mathrm{i}\theta}|{\uparrow}\rangle_1\}.$$

$$\left.\begin{array}{l}\{((1-s^2)^{1/2}|{\uparrow}\rangle_1 + \mathrm{i}s\mathrm{e}^{\mathrm{i}\theta}|{\downarrow}\rangle_1), (-\mathrm{i}s\mathrm{e}^{-\mathrm{i}\theta}|{\uparrow}\rangle_1 - (1-s^2)^{1/2}|{\downarrow}\rangle_1)\} \\ \{(-\mathrm{i}s|{\uparrow}\rangle_1 - (1-s^2)^{1/2}\mathrm{e}^{\mathrm{i}\theta}|{\downarrow}\rangle_1), ((1-s^2)^{1/2}\mathrm{e}^{-\mathrm{i}\theta}|{\uparrow}\rangle_1 + \mathrm{i}s|{\downarrow}\rangle_1)\}\end{array}\right. \qquad 0 \leqslant s < 1.$$

For $\theta = 0$ and $s = 0$ we have the special case of Bell's observable utilized by Bennett *et al* (1993).

The general solution for $N = 2$ requires a more complicated presentation. Therefore we omit it in this illustration.

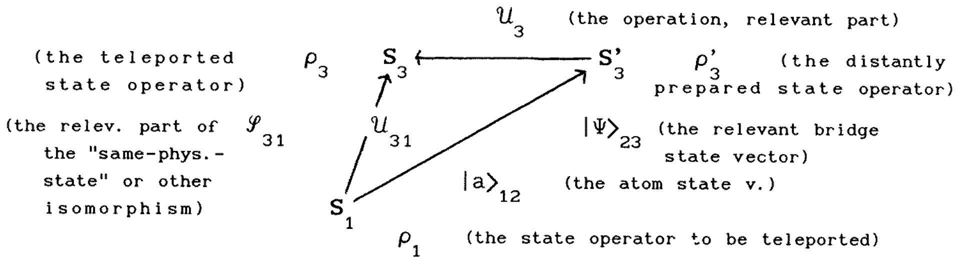

Figure 4.

8. Teleportation of proper mixtures

First, we generalize *statistical teleportation* to *general* (mixed or pure) *states*. We shall deal with state operators (statistical operators) instead of state vectors in this case.

One expects that generalizing the state vectors $|\psi\rangle_1, |\psi\rangle'_3$ $(= \mathcal{U}_{31}|\psi\rangle_1)$, and $|\psi\rangle_3$ $(= (\mathcal{U}_3 \circ \mathcal{U}_{31})|\psi\rangle_1 = \mathcal{I}_{31}|\psi\rangle_1)$ to state operators ρ_1, ρ'_3 and ρ_3 via the corresponding similarity transformations

$$\rho'_3\ (= \mathcal{U}_{31}\rho_1\mathcal{U}_{31}^{-1}) \qquad \rho_3\ (= (\mathcal{U}_3 \circ \mathcal{U}_{31})\rho_1(\mathcal{U}_3 \circ \mathcal{U}_{31})^{-1} = \mathcal{I}_{31}\rho_1\mathcal{I}_{31}^{-1})$$

one achieves statistical teleportation of the state operator.

We want to achieve figure 4.

One wonders what the counterpart of the restriction $|\psi\rangle_1 \in \boldsymbol{S}_1$ for the state operator ρ_1 is. The answer to this question is important because all that was stated about ρ_1 above cannot be expected to be valid in more generality than under the sought for restriction. We shall say that ρ_1 satisfying this restriction is 'suitable'.

Definition 11. A state operator ρ_1 will be called *suitable* if it satisfies one of the following four equivalent conditions.

(i) (*Viewing* ρ_1 *as one mixture*.) There exists a decomposition of ρ_1 into pure states, $\rho_1 = \sum_k w_k|\psi_k\rangle_1\langle\psi_k|_1$, such that $\forall k : |\psi_k\rangle_1 \in \boldsymbol{S}_1$.

(ii) (*Algebraic characterization*.) $P_1\rho_1 = \rho_1$.

(iii) (*Geometric characterization*.) The subspace $\boldsymbol{S}_1$ is invariant for ρ_1 and the latter reduces into zero in the orthocomplement $\boldsymbol{S}_1^{\perp}$.

(iv) (*Viewing* ρ_1 *as an arbitrary mixture of pure states*.) For every decomposition $\rho_1 = \sum_k w_k|\psi_k\rangle_1\langle\psi_k|_1$ of ρ_1 into pure states one has $\forall k : |\psi_k\rangle_1 \in \boldsymbol{S}_1$.

It will be proved in appendix A that the four restrictions on ρ_1 displayed in definition 11 are equivalent.

A plausibility argument for statistical teleportation of state operators goes as follows. Thinking of ρ_1 as of a mixture of pure states, e.g.

$$\rho_1 = \sum_k w_k|\psi_k\rangle_1\langle\psi_k|_1$$

$(\forall k : w_k > 0, \langle\psi_k|_1|\psi_k\rangle_1 = 1$, and $|\psi_k\rangle_1 \in \boldsymbol{S}_1, \sum_k w_k = 1)$, one imagines that an individual first subsystem is necessarily in one of the states $|\psi_k\rangle_1$, and it is thus teleported with the probability $\bar{w}$ (cf (26)) that does not depend either on k or w_k (cf (34)). This uniformity of $\bar{w}$ with respect to the above decomposition suggests that ρ_1 is teleported as a whole.

However, whenever ρ_1 is a mixed state, the above decomposition is necessarily nonunique and imagining ρ_1 as an actual physical mixture of pure states is open to doubt in quantum mechanics. Hence, it is desirable to have a rigorous argument that applies to statistical operators as *compact entities*. This is the usual procedure in quantum mechanics, and we proceed along these lines also for teleportation.

The first step, i.e. *distant preparation*, is crucial in statistical teleportation. To begin with, we generalize the most important results of distant-preparation theory (see section 2) to statistical operators. We return to the notation used in section 2.

Proposition 3. Let ρ_{nd} be a given general composite-system state operator (in $(\mathcal{H}_n \otimes \mathcal{H}_d)$), and $|b\rangle_n$ a given nearby-subsystem state vector. Then ideal measurement of the event $(|b\rangle_n\langle b|_n \otimes I_d)$ in the state ρ_{nd}, if the event does occur, brings about distant preparation of the distant subsystem in the state

$$\rho'_d \equiv p^{-1}\,\mathrm{Tr}_n\,\rho_{nd}(|b\rangle_n\langle b|_n \otimes I_d) \tag{56a}$$

where p is the probability of occurrence

$$p \equiv \mathrm{Tr}_{nd}\,\rho_{nd}(|b\rangle_n\langle b|_n \otimes I_d) = \langle b|_n\rho_n|b\rangle_n \qquad \rho_n \equiv \mathrm{Tr}_d\,\rho_{nd}. \tag{56b}$$

Proof. Ideal occurrence brings about a change of state given by the Lüders formula (Lüders 1951, Messiah 1961):

$$\rho_{nd} \to \rho'_{nd} \equiv p^{-1}(|b\rangle_n\langle b|_n \otimes I_d)\rho_{nd}(|b\rangle_n\langle b|_n \otimes I_d).$$

One has

$$\mathrm{Tr}_n[(E_n \otimes I_d)B_{nd}] = \mathrm{Tr}_n[B_{nd}(E_n \otimes I_d)]$$

where E_n is a projector in $\mathcal{H}_n$ and B_{nd} is an everywhere defined linear operator in $\mathcal{H}_{nd}$. The equality can be seen to hold if the partial trace is evaluated in a characteristic basis of E_n. Since $\rho'_d \equiv \mathrm{Tr}_n\,\rho'_{nd}$, the above commutation under the partial trace and the projector idempotency lead to (56*a*). □

Theorem 6. Any direct measurement of an event $(|b\rangle_n\langle b|_n \otimes I_d)$ in the state ρ_{nd} results in the distant preparation of the state operator ρ'_d given by (56*a*) as in the case of ideal measurement (and, of course, the probability p given by (56*b*) is the same).

Proof. Obtainable by direct generalization of that of theorem 1, in particular, by replacing $\langle\chi|_{nd}\ldots|\chi\rangle_{nd}$ in (5) by $(\mathrm{Tr}_{nd}\,\rho_{nd}\ldots)$. □

We now return to our investigation of teleportation.

Proposition 4. Let

$$\rho_1 = \sum_k w_k|\psi_k\rangle_1\langle\psi_k|_1 \tag{57a}$$

be a given decomposition into pure states of a state operator ρ_1 satisfying restriction (i) in definition 11. Let, further, $|\bar{a}\rangle_{12}$ be an atomic state vector in $(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Let, finally, a measurement of $(|\bar{a}\rangle_n\langle\bar{a}|_n \otimes I_d)$ in the three-subsystem composite-system state $(|\psi_k\rangle_1 \otimes |\bar{\Psi}\rangle_{23})$, if the event occurs, give by distant preparation the third-subsystem state $|\psi_k\rangle'_3$ for each value of k. Then, the same measurement in the state $(\rho_1 \otimes |\bar{\Psi}\rangle_{23}\langle\bar{\Psi}|_{23})$ in case of occurrence, gives by distant preparation the state ρ'_3 satisfying

$$\rho'_3 = \sum_k w_k|\psi_k\rangle'_3\langle\psi_k|'_3. \tag{57b}$$

Proof. According to theorem 6, one has

$$\rho_3' = (\bar{w})^{-1}\,\mathrm{Tr}_{12}(\rho_1 \otimes |\bar{\Psi}\rangle_{23}\langle\bar{\Psi}|_{23})(|\bar{a}\rangle_{12}\langle\bar{a}|_{12} \otimes I_3).$$

Substitution of decomposition (57*a*) gives further

$$\rho_3' = \sum_k w_k (\bar{w})^{-1} \langle\bar{a}|_{12}|\psi_k\rangle_1|\bar{\Psi}\rangle_{23}\langle\psi_k|_1\langle\bar{\Psi}|_{23}|\bar{a}\rangle_{12} = \sum_k w_k |\psi_k\rangle_3'\langle\psi_k|_3'$$

(cf theorem 1 and relation (4), as well as proposition 2). □

Theorem 7. If, under the assumptions of proposition 4, ρ_3' is the state operator to which ρ_1 gives rise by distant preparation, then one can write

$$\rho_3' = \mathcal{U}_{31}\rho_1\mathcal{U}_{31}^{-1}. \tag{58}$$

Proof. Follows immediately if one substitutes

$$\forall k : |\psi_k\rangle_3' = \mathcal{U}_{31}|\psi_k\rangle_1$$

in (57*b*) and one takes into account (57*a*). □

The two most important consequences are now obvious.

Corollary 20. Under the assumptions given in proposition 4, one has

$$\rho_3 \equiv (\mathcal{U}_3 \circ \mathcal{U}_{31})\rho_1(\mathcal{U}_3 \circ \mathcal{U}_{31})^{-1} = \mathcal{I}_{31}\rho_1\mathcal{I}_{31}^{-1}. \tag{59}$$

Corollary 21. If A_{12} is a complete observable defined in terms of atomic characteristic state vectors throughout like the one given by (45*a*), (45*b*), then its measurement in the state $(\rho_1 \otimes |\bar{\Psi}\rangle_{23}\langle\bar{\Psi}|_{23})$, where ρ_1 is a suitable state operator (cf definition 11), necessarily results in teleportation expressed by (59), where $\mathcal{U}_3$ and $\mathcal{U}_{31}$ have to be replaced by $\mathcal{U}_3^{(m)}$ and $\mathcal{U}_{31}^{(m)}$ respectively, m being the index of the atomic event $(|\bar{a}\rangle_{12}^{(m)}\langle\bar{a}|_{12}^{(m)} \otimes I_3)$ that did occur for the given individual system.

9. Teleportation of improper mixtures

We now assume that one more subsystem has entered the scene. We denote it by zero. Let ρ_{01} be an arbitrary state operator of the composite system $(0+1)$. The state operator ρ_1 $(\equiv \mathrm{Tr}_0\,\rho_{01})$ of subsystem 1 describes now an *improper mixture* (cf D'Espagnat 1976) unless ρ_{01} is uncorrelated (i.e. unless $\rho_{01} = \rho_0 \otimes \rho_1$, where $\rho_0 \equiv \mathrm{Tr}_1\,\rho_{01}$).

The *correlations* in a correlated state operator ρ_{01} are given in terms of the totality of *conditional state operators* $\{\rho_0(F_1)\text{: all } F_1\}$ acting in $\mathcal{H}_0$, where an arbitrary event (projector) F_1 for subsystem 1 is the condition. The state $\rho_0(F_1)$ of subsystem 0 comes about when $(I_0 \otimes F_1)$ (I_0 being the identity operator in $\mathcal{H}_0$) is measured in the state ρ_{01} and the event occurs (for more details see section 2 in Herbut 1986).

Theorem 8. Let

$$|\chi\rangle_{01} = \sum_i r_i^{1/2}|b_i\rangle_0|c_i\rangle_1 \tag{60}$$

be a state vector in $(\mathcal{H}_0 \otimes \mathcal{H}_1)$ given in a Schmidt biorthogonal expansion (cf definition 4) and such that the state operator of subsystem 1, i.e. ρ_1 $(\equiv \mathrm{Tr}_0\,|\chi\rangle_{01}\langle\chi|_{01})$, satisfies the suitability restriction (cf definition 11). Let, further, $|\bar{\Psi}\rangle_{23}$ be a given bridge state vector, and let $|\bar{a}\rangle_{12}$ be an atomic state vector. Finally, let the event $(I_0 \otimes |\bar{a}\rangle_{12}\langle\bar{a}|_{12} \otimes I_3)$ be measured in the state $|\chi\rangle_{01}|\bar{\Psi}\rangle_{23}$. Then, if the event occurs, *distant preparation*, i.e. the first step of statistical teleportation, brings about a state vector $|\chi\rangle'_{03}$ so that

$$|\chi\rangle'_{03} = (I_0 \otimes \mathcal{U}_{31})|\chi\rangle_{01} \tag{61}$$

where $\mathcal{U}_{31}$ is the unitary isomorphism mapping $\boldsymbol{S}_1$ onto $\boldsymbol{S}'_3$ that is determined by $|\bar{a}\rangle_{12}$ and $|\bar{\Psi}\rangle_{23}$ (cf (6*a*) and (6*b*)).

Proof. Utilizing the basic distant-preparation formula (4) (cf also Theorem 1), and taking subsystem $(1+2)$ as the nearby and $(0+3)$ as the distant subsystem, one has

$$\begin{aligned}|\chi\rangle'_{03} &= (\bar{w})^{-1/2}\langle\bar{a}|_{12}\left(\sum_i r_i^{1/2}|b_i\rangle_0|c_i\rangle_1\right)|\bar{\Psi}\rangle_{23}\\ &= \sum_i r_i^{1/2}|b_i\rangle_0[(\bar{w})^{-1/2}\langle\bar{a}|_{12}|c_i\rangle_1)|\bar{\Psi}\rangle_{23}] = \sum_i r_i^{1/2}|b_i\rangle_0(\mathcal{U}_{31}|c_i\rangle_1)\\ &= (I_0\otimes\mathcal{U}_{31})\left(\sum_i r_i^{1/2}|b_i\rangle_0|c_i\rangle_1\right) = (I_0\otimes\mathcal{U}_{31})|\chi\rangle_{01}.\end{aligned}$$

Use has been made of (6*a*) and (6*b*) and of the fact that $\bar{w}$ does not depend on the state vector from $\boldsymbol{S}_1$. □

Corollary 22. Under the assumptions of theorem 8, *statistical teleportation* gives

$$|\chi\rangle_{03} = (I_0 \otimes \mathcal{U}_3) \circ (I_0 \otimes \mathcal{U}_{31})|\chi\rangle_{01} = (I_0 \otimes \mathcal{I}_{31})|\chi\rangle_{01}. \tag{62}$$

Proof. Obvious. □

Corollary 23. The *correlations* in a teleported state vector $|\chi\rangle_{01}$ are *preserved* in the sense that we have the Schmidt biorthogonal expansions

$$|\chi\rangle'_{03} = (I_0 \otimes \mathcal{U}_{31})|\chi\rangle_{01} = \sum_i r_i^{1/2}|b_i\rangle_0(\mathcal{U}_{31}|c_i\rangle_1)_3 \tag{63}$$

$$|\chi\rangle_{03} = (I_0 \otimes \mathcal{I}_{31})|\chi\rangle_{01} = \sum_i r_i^{1/2}|b_i\rangle_0(\mathcal{I}_{31}|c_i\rangle_1)_3 \tag{64}$$

essentially unchanged with respect to (60), i.e. unchanged in $\mathcal{H}_0$.

Proof. Follows immediately from (61) and (62). □

Theorem 9. Let $|\bar{\Psi}\rangle_{23}$ be a given bridge state vector, let $|\bar{a}\rangle_{12}$ be an atomic state vector, and let the event $(I_0 \otimes |\bar{a}\rangle_{12}\langle\bar{a}|_{12} \otimes I_3)$ be measured in the state given by the state operator $\rho_{01} \otimes (|\bar{\Psi}\rangle_{23}\langle\bar{\Psi}|_{23})$, where ρ_{01} is an arbitrary state operator in $(\mathcal{H}_0 \otimes \mathcal{H}_1)$, but such that the state operator of subsystem 1, i.e. ρ_1 ($\equiv \mathrm{Tr}_0\, \rho_{01}$), satisfies the suitability restriction (definition 11). Then, in case of occurrence, the distantly prepared state operator ρ'_{03} is obtained from ρ_{01} as follows

$$\rho'_{03} = (I_0 \otimes \mathcal{U}_{31})\rho_{01}(I_0 \otimes \mathcal{U}_{31})^{-1} \tag{65}$$

where $\mathcal{U}_{31}$ is the unitary isomorphism mapping $\boldsymbol{S}_1$ onto $\boldsymbol{S}'_3$ that is determined by $|\bar{a}\rangle_{12}$ and $|\bar{\Psi}\rangle_{23}$ (cf (6*a*) and (6*b*)).

Proof. Taking the subsystem $(1+2)$ as the nearby and $(0+3)$ as the distant subsystem, according to theorem 6 distant preparation gives

$$\begin{aligned} \rho'_{03} &= (\bar{w})^{-1}\,\mathrm{Tr}_{12}\{[\rho_{01}\otimes(|\bar{\Psi}\rangle_{23}\langle\bar{\Psi}|_{23})][I_0\otimes|\bar{a}\rangle_{12}\langle\bar{a}|_{12}\otimes I_3]\} \\ \bar{w} &= \mathrm{Tr}_{0123}\{[\rho_{01}\otimes(|\bar{\Psi}\rangle_{23}\langle\bar{\Psi}|_{23})][I_0\otimes|\bar{a}\rangle_{12}\langle\bar{a}|_{12}\otimes I_3]\}. \end{aligned} \tag{66}$$

Let

$$\rho_{01} = \sum_k w_k|\chi_k\rangle_{01}\langle\chi_k|_{01} \tag{67}$$

be a decomposition of ρ_{01} into pure states. Replacing this in (66), using the fact that $\bar{w}$ does not depend on $|\chi_k\rangle_{01}$, and utilizing (61), we derive

$$\rho'_{03} = \sum_k w_k(I_0\otimes\mathcal{U}_{31})|\chi_k\rangle_{01}\langle\chi_k|_{01}(I_0\otimes\mathcal{U}_{31})^{-1}$$

which, on account of (67), implies (65). □

Corollary 24. Under the assumptions of theorem 9, the statistically teleported state operator can be written

$$\rho_{03} = [(I_0\otimes\mathcal{U}_3)\circ(I_0\otimes\mathcal{U}_{31})]\rho_{01}[(I_0\otimes\mathcal{U}_3)\circ(I_0\otimes\mathcal{U}_{31})]^{-1} = (I_0\otimes I_{31})\rho_{01}(I_0\otimes I_{31})^{-1}. \tag{68}$$

Proof. Obvious. □

Theorem 10. Let F_1 be an arbitrary subprojector of P_1, i.e. such that $F_1P_1 = F_1$ is valid (in view of P_1, cf definition 11). Then *the conditional state operator* in $\mathcal{H}_0$, i.e.

$$\rho_0(F_1) \equiv p^{-1}\,\mathrm{Tr}_1[\rho_{01}(I_0\otimes F_1)] \tag{69a}$$

where

$$p \equiv \mathrm{Tr}_{01}[\rho_{01}(I_0\otimes F_1)] \tag{69b}$$

is *preserved* in the sense that

$$\rho_0(F_1) = \rho_0(\mathcal{U}_{31}F_1\mathcal{U}_{31}^{-1}) \tag{70a}$$

$$= \rho_0(\mathcal{I}_{31}F_1\mathcal{I}_{31}^{-1}). \tag{70b}$$

Proof. Denoting by p' the probability that $[I_0\otimes(\mathcal{U}_{31}F_1\mathcal{U}_{31}^{-1})]$ occurs in the state $\rho'_{03'}$ and utilizing (65), one has

$$\begin{aligned} \rho_0(\mathcal{U}_{31}F_1\mathcal{U}_{31}^{-1}) &\equiv (p')^{-1}\mathrm{Tr}_3\{\rho'_{03}[I_0\otimes(\mathcal{U}_{31}F_1\mathcal{U}_{31}^{-1})]\} \\ &= (p')^{-1}\,\mathrm{Tr}_3\{[(I_0\otimes\mathcal{U}_{31})\rho_{01}(I_0\otimes\mathcal{U}_{31})^{-1}][(I_0\otimes\mathcal{U}_{31})(I_0\otimes F_1)(I_0\otimes\mathcal{U}_{31})^{-1}]\} \\ &= (p')^{-1}\,\mathrm{Tr}_3[(I_0\otimes\mathcal{U}_{31})\rho_{01}(I_0\otimes F_1)(I_0\otimes\mathcal{U}_{31})^{-1}] = (p')^{-1}\,\mathrm{Tr}_1[\rho_{01}(I_0\otimes F_1)]. \end{aligned}$$

Using (65) again, one obtains

$$\begin{aligned} p' &\equiv \mathrm{Tr}_{03}\{\rho'_{03}[I_0\otimes(\mathcal{U}_{31}F_1\mathcal{U}_{31}^{-1})]\} \\ &= \mathrm{Tr}_{03}\{(I_0\otimes\mathcal{U}_{31})\rho_{01}(I_0\otimes\mathcal{U}_{31}^{-1})[(I_0\otimes\mathcal{U}_{31})(I_0\otimes F_1)(I_0\otimes\mathcal{U}_{31})^{-1}]\} \\ &= \mathrm{Tr}_{03}\{(I_0\otimes\mathcal{U}_{31})\rho_{01}(I_0\otimes F_1)(I_0\otimes\mathcal{U}_{31})^{-1}]\} = \mathrm{Tr}_{01}[\rho_{01}(I_0\otimes F_1)] = p. \end{aligned}$$

This establishes the validity of (70a). The proof of (70b) is analogous using (68) instead of (65). □

Corollary 25. If an observable A_{12} is suitable for deterministic teleportation, i.e. if it is complete and all its characteristic vectors are atomic state vectors, then all the results of this section are valid for deterministic teleportation via measurement of A_{12}.

Proof. Follows from the fact that, whatever the result in the measurement of A_{12}, statistical teleportation takes place. □

10. Concluding results and discussion

10.1. Why linearity or, equivalently, unitarity?

Assuming linearity of $\mathcal{U}_{31}$ (distant preparation), as we did in our derivation of the central theorem, and taking into account the fact that $\mathcal{U}_{31}$ preserves the norm, one concludes that it is unitary. Besides, since $\mathcal{U}_3 = \mathcal{I}_{31} \circ \mathcal{U}_{31}^{-1}$ (cf figure 2), and since $\mathcal{I}_{31}$ is a unitary isomorphism, also $\mathcal{U}_3$ is unitary.

One wonders if this is necessary. It might be conceivable that $\mathcal{U}_{31}$ and $\mathcal{U}_3$ violate linearity, but in way that cancels out in consecutive application (like e.g. in case of antiunitary operators). We show now that under a few assumptions that we consider essential for statistical teleportation, $\mathcal{U}_3$ and $\mathcal{U}_{31}$ *must be unitary*.

Our *assumptions* are as follows.

(i) There exists a physical system, we call it (the distant) 'laboratory', and denote it by 'L', such that the composite system $(3 + L)$ is dynamically closed during teleportation. Hence, the evolution of the system $(3+L)$ during the second step of teleportation is governed by a unitary operator $\mathcal{U}_{3L}$.

(ii) The state of the laboratory at the moment when the first step, i.e. distant preparation, is completed, is described by one and the same state vector $|\Psi\rangle'_L$ (or state operator ρ'_L) independently of the distantly prepared state vector $|\psi\rangle'_3$ ($\equiv \mathcal{U}_{31}|\psi\rangle_1$, $|\psi\rangle_1 \in \boldsymbol{S}_1$).

(iii) Besides the initial state vector $|\psi\rangle'_3|\Psi\rangle'_L$ for the second step of teleportation also the final state vector $|\psi\rangle_3|\Psi\rangle_L$ has to be uncorrelated for each $|\psi\rangle'_3$ ($\in \boldsymbol{S}'_3 \equiv \mathcal{U}_{31}\boldsymbol{S}_1$).

(iv) Both $\mathcal{U}_{31}$ and $\mathcal{U}_3$ have to be bijective (because so is the unitary isomorphism $\mathcal{I}_{31} = \mathcal{U}_3 \circ \mathcal{U}_{31}$).

(v) If $|\psi\rangle'_3, |\bar{\psi}\rangle'_3 \in \boldsymbol{S}'_3$, and $\langle\psi|'_3|\bar{\psi}\rangle'_3 = 0$, then the teleported state vectors cannot be collinear, i.e. $|\psi\rangle_3 \neq \mathrm{e}^{\mathrm{i}\lambda}|\bar{\psi}\rangle_3$, $\lambda \in \boldsymbol{R}$.

Violation of (v) would also violate the requirement that teleportation take distinct pure states into distinct pure states, or else, distant preparation would have to take two state vectors $|\psi\rangle_1, |\bar{\psi}\rangle_1 \in \boldsymbol{S}_1$ representing the same pure state ($|\bar{\psi}\rangle_1 = \mathrm{e}^{\mathrm{i}\omega}|\psi\rangle_1, \omega \in \boldsymbol{R}$) into orthogonal state vectors $|\psi\rangle'_3, |\bar{\psi}\rangle'_3 \in \boldsymbol{S}_3$. This is not possible, because it is obvious from the distant-preparation formula

$$|\psi\rangle'_3 \equiv (\bar{w})^{-1/2}\langle\bar{a}|_{12}|\psi\rangle_1|\bar{\psi}\rangle_{23}$$

(cf (6a)) that one has necessarily $|\bar{\psi}\rangle'_3 = \mathrm{e}^{\mathrm{i}\omega}|\psi\rangle'_3, \omega \in \boldsymbol{R}$, in our case.

The unitarity of $\mathcal{U}_{3L}$, in conjunction with assumptions (ii)–(iv), implies that one can fix a state vector $|\Psi\rangle_L$ of the 'laboratory' that is independent of $|\psi\rangle'_3(\in \boldsymbol{S}'_3)$, and define an operator $\mathcal{U}_3$ mapping $\boldsymbol{S}'_3$ onto $\boldsymbol{S}_3$ that is almost-linear. This is proved in appendix B.

Further, on account of the almost-linearity of the operator $\mathcal{U}_3$, so is $\mathcal{U}_3^{-1}$, and $\mathcal{U}_{31}$ $(= \mathcal{U}_3^{-1} \circ \mathcal{I}_{31})$. This almost-linearity of $\mathcal{U}_{31}$ is proved in appendix C.

The operator $\mathcal{U}_{31}$ turns out to be 'linear with respect to' a fixed set of vectors. This is proved in appendix D. This is sufficient for it to be linear. This is proved in appendix E.

Since $\mathcal{U}_{31}$ takes state vectors into state vectors and it is linear, it is unitary; and so is $\mathcal{U}_3$ $(= \mathcal{I}_{31} \circ \mathcal{U}_{31}^{-1})$.

Finally, it is hard to imagine a macroscopic system like the one we call the (distant) 'laboratory' in a pure state because practically it is almost impossible to bring a system of great complexity into such a state. Hence, it is desirable to assume that the laboratory is in a mixed state. We generalize our proof to the case when the state vectors $|\Psi\rangle_L'$ and $|\Psi\rangle_L$ are replaced by state operators ρ_L' and ρ_L in appendix F. (In the proof we restrain from the dubious physical assumption that the individual laboratory must be in some unknown pure state.)

10.2. Is there any intermediary case between statistical and deterministic teleportation?

One may have an observable B_{12} that has some nondegenerate characteristic values with atomic state vectors as the corresponding characteristic vectors (teleporting results), but that also has degenerate or nondegenerate not teleporting characteristic values. Then, after measuring this observable, if one obtains a teleporting result, one 'phones' the distant laboratory and one informs it which result it is. There one applies the corresponding operation and the (statistical) teleportation is achieved (in a subensemble).

This is, actually, a more intricate form of statistical teleportation.

10.3. How complete is our discussion on statistical and deterministic teleportation?

To end our investigation in a self-critical way, let us sum up what has been done emphasizing what has to be done in further research.

Statistical teleportation has been treated in this work in a rather complete fashion. The general form of an atomic state vector $|\bar{a}\rangle_{12}$ has been found via (33) and (34), and the set of *all* solutions in relevant atom state vectors $|a\rangle_{12}$ has been classified in the following three ways.

(i) Classification in two steps. In the first step we consider all atom state vectors $|a\rangle_{12}$ determining one and the same intermediary space $\boldsymbol{S}_3'$ (cf (6*a*,*b*)), which then, in turn, determines $\boldsymbol{S}_2$ (cf definition 9), and, in the second step through arbitrary ON bases $\{|i\rangle_1 : i = 1, 2, \ldots, N\}$ in $\boldsymbol{S}_1$ in the Schmidt biorthogonal expansion of $|a\rangle_{12}$:

$$|a\rangle_{12} = \sum_i w^{1/2} r_i^{-1/2} |i\rangle_1 |i_2\rangle$$

where the ON basis $\{|i\rangle_2 : i = 1, 2, \ldots, N\}$ spanning $\boldsymbol{S}_2$ is an arbitrary fixed characteristic subbasis of ρ_2^B (spanning its range).

(ii) The first step is the same as in (i), and in the second classification takes place through arbitrary inverse correlation operators $\mathcal{U}_a^{-1}$ (cf (15*b*)):

$$|a\rangle_{12} = \sum_{i=1}^{N} w^{1/2} r_i^{-1/2} [(\mathcal{U}_a)^{-1} |i\rangle_2]_1 \otimes |i\rangle_2.$$

(iii) After the first step as in (i), we have classification through arbitrary unitary operators $\mathcal{U}_1$ and $\boldsymbol{S}_1$:

$$|a\rangle_{12} = (\mathcal{U}_1 \otimes I_2) |a^{(0)}\rangle_{12}$$

where $|a^{(0)}\rangle_{12}$ is a fixed atom state vector (cf corollary 15).

What we have not done in this study is a clarification which spaces $\boldsymbol{S}_3'$ $(\subseteq R[(\bar{\rho}_3^B)]^{1/2})$ do correspond to some atom state vectors $|a\rangle_{12}$ and which do not. Then, the 'good' $\boldsymbol{S}_3'$

serve in a precise way as the first step of classification, and (taking $\boldsymbol{S}_3'$ as the third input entity) the above classifications are complete.

For deterministic teleportation we have obtained necessary and sufficient conditions, but without a general solution. It is not clear whether, for an arbitrary given entangled bridge state vector $|\bar{\Psi}\rangle_{23}$, and for a given finite-dimensional subspace $\boldsymbol{S}_1$ $(\subseteq \mathcal{H}_1)$ there does necessarily exist a deterministically teleporting observable A_{12}. If there exists one, it is not clear how one can classify *all* such observables A_{12}.

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Appendix A

In connection with definition 11 (on suitability of a state operator), we prove now:

Lemma A.1. Let $\boldsymbol{S}_1$ $(\subseteq \mathcal{H}_1)$ be a subspace, and let the projector P_1 project $\mathcal{H}_1$ onto it. The following four properties of a state operator ρ_1 are equivalent to each other.

(i) There exists a decomposition of ρ_1 into pure states, $\rho_1 = \sum_k w_k |\psi_k\rangle_1 \langle\psi_k|_1$, such that $\forall k : |\psi_k\rangle_1 \in \boldsymbol{S}_1$.

(ii) $P_1\rho_1 = \rho_1$.

(iii) The subspace $\boldsymbol{S}_1$ is invariant for ρ_1 and the latter reduces into zero in its orthocomplement $\boldsymbol{S}_1^{\perp}$.

(iv) For every decomposition of ρ_1 into pure states, each state vector obtained in the decomposition (cf (i)) belongs to $\boldsymbol{S}_1$.

Proof. The relation $|\psi_k\rangle_1 \in \boldsymbol{S}_1$ is equivalent to $P_1|\psi_k\rangle_1 = |\psi_k\rangle_1$. Hence, (i) obviously implies (ii). Adjoining the latter, we see that P_1 and ρ_1 commute. Hence, $\boldsymbol{S}_1$ and $\boldsymbol{S}_1^{\perp}$ are invariant for ρ_1. If $|\phi\rangle_1 \in \boldsymbol{S}_1^{\perp}$, then, also $(\rho_1|\phi\rangle_1) \in \boldsymbol{S}_1^{\perp}$. On the other hand, as a consequence of (ii), $P_1(\rho_1|\phi\rangle_1) = (\rho_1|\phi\rangle_1)$. Since $\boldsymbol{S}_1^{\perp} = (R(P_1))^{\perp}$, the last two relations are possible only if $(\rho_1|\phi\rangle_1) = 0$. Thus, (ii) implies (iii).

We assume that (iii) is valid, and we take an orthonormal subbasis $\{|\phi_j\rangle_1 : \forall j\}$ spanning $\boldsymbol{S}_1^{\perp}$. Let $\rho_1 = \sum_k w_k |\psi_k\rangle_1 \langle\psi_k|_1$ be an arbitrary decomposition of ρ_1 into pure states. Then,

$$\forall j : 0 = \langle\phi_j|\rho_1|\phi_j\rangle_1 = \sum_k w_k \langle\phi_j|_1|\psi_k\rangle_1 \langle\psi_k|_1|\phi_j\rangle_1.$$

Hence, $\forall j, \forall k : \langle\phi_j|\psi_k\rangle_1 = 0$, and this implies that $\forall k : |\psi_k\rangle_1 \in \boldsymbol{S}_1$. Thus, (iii) implies (iv). Finally, (i) is a consequence of (iv) because there always exists the spectral decomposition of ρ_1 into characteristic vectors (and w_k are then the corresponding characteristic values). $\square$

Appendix B

Lemma B.1. One can fix a state vector $|\Psi\rangle_L$ of the 'laboratory' that is independent of $|\psi\rangle_3'$ $(\in \boldsymbol{S}_3')$, and such that

$$\forall |\psi\rangle_3' \in \boldsymbol{S}_3': \qquad \mathcal{U}_{3L}(|\psi\rangle_3' \otimes |\Psi\rangle_L') \equiv (\mathcal{U}_3|\psi\rangle_3') \otimes |\psi\rangle_L$$

where $\mathcal{U}_3$ is almost-linear in the sense that for a certain choice of an ON basis $\{|k\rangle_3' : \forall k\}$ spanning $\boldsymbol{S}_3'$ one has

$$|\psi\rangle_3' = \sum_k \alpha_k |k\rangle_3' \Rightarrow \mathcal{U}_3|\psi\rangle_3' = \sum_k \alpha_k \mathrm{e}^{\mathrm{i}\lambda_k}(\mathcal{U}_3|k\rangle_3') \tag{B.1}$$

($\lambda_k \in \mathbf{R}$, and they depend on the corresponding $|k\rangle_3'$, but not on $|\psi\rangle_3'$).

Before we prove lemma B.1, we insert an auxiliary result.

Lemma B.2. If $|\psi\rangle_3', |\bar{\psi}\rangle_3' \in \boldsymbol{S}_3'$, and $|\psi\rangle_3', |\bar{\psi}\rangle_3'$ are state vectors orthogonal to each other, then $|\Psi\rangle_L = \mathrm{e}^{\mathrm{i}\lambda}|\bar{\Psi}\rangle_L, \lambda \in \mathbf{R}$, where $\mathcal{U}_{3L}(|\psi\rangle_3'|\Psi\rangle_3') = |\psi\rangle_3|\Psi\rangle_L$ and $\mathcal{U}_{3L}(|\bar{\psi}\rangle_3|\Psi\rangle_3') = |\bar{\psi}\rangle_3|\bar{\Psi}\rangle_L$.

Proof. Since the state vectors $|\psi\rangle_3', |\bar{\psi}\rangle_3'$ are orthogonal, so are $|\psi\rangle_3'|\Psi\rangle_L'$ and $|\bar{\psi}\rangle_3'|\Psi\rangle_L'$. Then, on account of the unitarity of $\mathcal{U}_{3L}$ (cf assumption (i)), so are also $|\psi\rangle_3|\Psi\rangle_L (= \mathcal{U}_{3L}(|\psi\rangle_3'|\Psi\rangle_L'))$ and $|\bar{\psi}\rangle_3|\bar{\Psi}\rangle_L (= \mathcal{U}_{3L}(|\bar{\psi}\rangle_3'|\Psi\rangle_L'))$. Hence, either $\langle\Psi|_L|\bar{\Psi}\rangle_L = 0$ or $\langle\psi|_3|\bar{\psi}\rangle_3 = 0$.

(a) Let $\langle\Psi|_L|\bar{\Psi}\rangle_L = 0$. Then, due to assumptions (i) and (ii), one has:

$$\mathcal{U}_{3L}(|\psi\rangle_3'|\Psi\rangle_L' + |\bar{\psi}\rangle_3'|\Psi\rangle_L') = |\psi\rangle_3|\Psi\rangle_L + |\bar{\psi}\rangle_3|\bar{\Psi}\rangle_L. \tag{B.2}$$

On the other hand, LHS $= \mathcal{U}_{3L}[(|\psi\rangle_3' + |\bar{\psi}\rangle_3') \otimes |\Psi\rangle_L']$, and owing to assumption (iii)

$$\text{LHS} = p|x\rangle_3|\Psi_x\rangle_L \tag{B.3}$$

where p is the norm of the LHS. Relation (B.2) then gives $p^2 = 2$. Further, (B.2) and (B.3) imply

$$|\Psi_x\rangle_L = 2^{-1/2}(\langle x|_3|\psi\rangle_3)|\Psi\rangle_L + (\langle x|_3|\bar{\psi}\rangle_3)|\bar{\Psi}\rangle_L. \tag{B.4}$$

Since in (B.4) all kets are vectors of norm 1 and by assumption (a) $\langle\Psi|_L|\bar{\Psi}\rangle_L = 0$, necessarily

$$|\langle x|_3|\psi\rangle_3| = |\langle x|_3|\bar{\psi}\rangle_3| = 1$$

or

$$\mathrm{e}^{\mathrm{i}\mu}|\psi\rangle_3 = |x\rangle_3 = \mathrm{e}^{\mathrm{i}\omega}|\bar{\psi}\rangle_3 \qquad \mu, \omega \in \mathbf{R}.$$

Owing to the assumption $\langle\psi|_3'|\bar{\psi}\rangle_3' = 0$, and to assumption (v) in the text, this is not possible.

(b) Let $\langle\psi|_3|\bar{\psi}\rangle_3 = 0$. Since the two tensor factors play symmetrical roles, one proves in analogy with (a) that

$$\mathrm{e}^{\mathrm{i}\mu}|\Psi\rangle_L = |\Psi_x\rangle_L = \mathrm{e}^{\mathrm{i}\omega}|\bar{\Psi}\rangle_L \qquad \mu, \omega \in \mathbf{R}.$$

□

Proof. Proof of lemma B.1. We assume that we have two arbitrary state vectors $|\psi\rangle_3', |1\rangle_3' \in \boldsymbol{S}_3'$, and we complete the second one into an orthonormal basis $\{|k\rangle_3' : k = 1, 2, \ldots, N\}$ spanning $\boldsymbol{S}_3'$. Finally, let $|\psi\rangle_3' = \sum_k \alpha_k |k\rangle_3'$. Then

$$\mathcal{U}_{3L}(|\psi\rangle_3'|\Psi\rangle_L') = |\psi\rangle_3|\Psi\rangle_L$$

and, on the other hand, since $\mathcal{U}_{3L}$ is linear, utilizing lemma B.2, one can write

$$\text{LHS} = \sum_k \alpha_k \mathcal{U}_{3L}(|k\rangle_3'|\Psi\rangle_L') = \sum_k \alpha_k |k\rangle_3|\Psi_k\rangle_L = \left(\alpha_1|1\rangle_3 + \sum_{k=2}^N \alpha_k \mathrm{e}^{\mathrm{i}\lambda_k}|k\rangle_3\right)|\Psi_1\rangle_L.$$

Hence, we can define

$$|\Psi\rangle_L \equiv |\Psi_1\rangle_L \qquad |\psi\rangle_3 \equiv \mathcal{U}_3|\psi\rangle_3' \equiv \sum_{k=1}^{N} \alpha_k \mathrm{e}^{\mathrm{i}\lambda_k}|k\rangle_3$$

with

$$\forall k: \qquad \mathcal{U}_{3L}(|k\rangle_3'|\Psi\rangle_L') = |k\rangle_3|\Psi\rangle_L \equiv (\mathcal{U}_3|k\rangle_3')|\Psi\rangle_L$$

and $\lambda_1 \equiv 0$. □

Appendix C

The inverse $\mathcal{U}_3^{-1}$ exists due to assumption (iv) (cf section 10.1). First, we prove that it is almost-linear with respect to the set of vectors $\{|k\rangle_3 \equiv \mathcal{U}_3|k\rangle_3' : \forall k\}$ in the span of these vectors. Utilizing (B.1), one obtains:

$$|\psi\rangle_3 \equiv \sum_k \beta_k|k\rangle_3 = \sum_k \beta_k(\mathcal{U}_3|k\rangle_3') = \mathcal{U}_3\left(\sum_k \beta_k \mathrm{e}^{-\mathrm{i}\lambda_k}|k\rangle_3'\right)$$

hence,

$$\mathcal{U}_3^{-1}|\psi\rangle_3 = \sum_k \beta_k \mathrm{e}^{-\mathrm{i}\lambda_k}(\mathcal{U}_3^{-1}|k\rangle_3). \tag{C.1}$$

Next, we prove that $\mathcal{U}_{31}$ $(= \mathcal{U}_3^{-1} \circ \mathcal{I}_{31})$ is almost-linear with respect to the set of vectors $\{|k\rangle_1 \equiv \mathcal{U}_{31}^{-1}|k\rangle_3' : \forall k\}$ in their span. The inverse $\mathcal{U}_{31}^{-1}$ exists on account of assumption (iv). Let $|\psi\rangle_1 \equiv \sum_k \beta_k|k\rangle_1$. Then, owing to (C.1),

$$\mathcal{U}_{31}|\psi\rangle_1 = (\mathcal{U}_3^{-1} \circ \mathcal{I}_{31})|\psi\rangle_1 = \mathcal{U}_3^{-1}\left(\sum_k \beta_k|k\rangle_3\right) = \sum_k \beta_k \mathrm{e}^{-\mathrm{i}\lambda_k}|k\rangle_3' = \sum_k \beta_k \mathrm{e}^{-\mathrm{i}\lambda_k}(\mathcal{U}_{31}|k\rangle_1). \tag{C.2}$$

□

Appendix D

By a variation of the necessity-part of the proof of lemma 8, we first prove that $\mathcal{U}_{31}$ satisfying (C.2) is linear with respect to the set of vectors $\{|k\rangle_1 \equiv \mathcal{U}_{31}^{-1}|k\rangle_3' : \forall k\}$ i.e. that

$$|\psi\rangle_1 \equiv \sum_k \beta_k|k\rangle_1 \Rightarrow \mathcal{U}_{31}|\psi\rangle_1 = \sum_k \beta_k(\mathcal{U}_{31}|k\rangle_1). \tag{D.1}$$

Let the probabilities of distant preparation be w and $\{w_k : \forall k\}$ respectively, i.e.

$$\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}) = w^{1/2}|\psi\rangle_3' \tag{D.2a}$$

$$\forall k: \qquad \langle a|_{12}(|k\rangle_1|\Psi\rangle_{23}) = w_k^{1/2}|k\rangle_3'. \tag{D.2b}$$

Relation (D.2a) can be rewritten in the form

$$\langle a|_{12}(|\psi\rangle_1|\Psi\rangle_{23}) \equiv w^{1/2}(\mathcal{U}_{31}|\psi\rangle_1).$$

Replacing here $|\psi\rangle_1$ in its expanded form (cf (D.1)), and utilizing (D.2b) and (C.2), one obtains

$$\sum_k \beta_k w_k^{1/2}|k\rangle_3' = w^{1/2}\sum_k \beta_k \mathrm{e}^{-\mathrm{i}\lambda_k}|k\rangle_3'. \tag{D.3}$$

Since $\{|k\rangle_3' : \forall k\}$ was taken to be an ON basis in S_3' (cf appendix B), (D.3) implies

$$\forall k : \beta_k w_k^{1/2} = \beta_k w^{1/2} e^{-i\lambda_k}. \tag{D.4}$$

Choosing $|\psi\rangle_1$ so that all expansion coefficients are nonzero (cf (D.1)), (D.4) entails: $\forall k : e^{-i\lambda k} = 1$. Substituting this in (C.2), we derive the claim of (D.1).

Appendix E

Lemma E.1. If an operator $\mathcal{U}_{31}$, mapping S_1 into S_3', is 'linear with respect to' a set of vectors $\{|k\rangle_1 : \forall k\}$, then it is linear in the span of these vectors.

Proof. Let $|\psi\rangle_1$ and $\{|\psi_j\rangle_1 : j = 1, 2, \ldots, J\}$ be in the span of the above vectors, i.e. let it be possible to expand them (at least in one way):

$$|\psi\rangle_1 = \sum_k \beta_k |k\rangle_1 \tag{E.1a}$$

$$\forall_j : \qquad |\psi_j\rangle_1 = \sum_k \gamma_{jk} |k\rangle_1. \tag{E.1b}$$

Besides, let

$$|\psi\rangle_1 = \sum_{j=1}^{J} \alpha_j |\psi_j\rangle_1. \tag{E.2}$$

Substituting (E.1) in (E.2), we further have

$$|\psi\rangle_1 = \sum_{j=1}^{J} \sum_k \alpha_j \gamma_{jk} |k\rangle_1.$$

On account of the 'linearity with respect to' the set $\{|k\rangle_1 : \forall k\}$, one has

$$\mathcal{U}_{31}|\psi\rangle_1 = \sum_{j=1}^{J} \sum_k \alpha_j \gamma_{jk} (\mathcal{U}_{31}|k\rangle_1). \tag{E.3}$$

On the other hand, applying $\mathcal{U}_{31}$ to (E.1b), one obtains

$$\mathcal{U}_{31}|\psi_j\rangle_1 = \sum_k \gamma_{jk} (\mathcal{U}_{31}|k\rangle_1).$$

Replacing this in (E.3), one arrives at

$$\mathcal{U}_{31}|\psi\rangle_1 = \sum_{j=1}^{J} \alpha_j (\mathcal{U}_{31}|\psi_j\rangle_1).$$

In view of the arbitrariness of the linear combination (E.2), this proves linearity. □

Appendix F

Lemma F.1. The final state operator ρ_L of the 'laboratory' is independent of the distantly-prepared state vector $|\psi\rangle'_3$.

Proof. The assumptions on the second step of statistical teleportation (see section 10.1) allow us to write

$$|\psi\rangle_3\langle\psi|_3 \otimes \rho_L = \mathcal{U}_{3L}(|\psi\rangle'_3\langle\psi|'_3 \otimes \rho'_L)\mathcal{U}^\dagger_{3L}$$

where ρ_L might depend on $|\psi\rangle'_3$. Let

$$\rho'_L = \sum_i w_i |\Psi_i\rangle'_L\langle\Psi_i|'_L$$

be a spectral form of ρ'_L. Substitution of the latter relation in the former gives

$$|\psi\rangle_3\langle\psi|_3 \otimes \rho_L = \sum_i w_i \mathcal{U}_{3L}(|\psi\rangle'_3\langle\psi|'_3 \otimes |\Psi_i\rangle'_L\langle\Psi_i|'_L)\mathcal{U}^\dagger_{3L}. \tag{F.1}$$

Since for any value of i, $|\Psi_i\rangle'_L$ is independent of $|\psi\rangle'_3$ (because so is ρ'_L by assumption (ii) in the text), according to lemma B.1, we have

$$\forall i : \mathcal{U}_{3L}(|\psi\rangle'_3\langle\psi|'_3 \otimes |\Psi_i\rangle'_L\langle\Psi_i|'_L)\mathcal{U}^\dagger_{3L} = (|\psi\rangle_3\langle\psi|_3 \otimes |\Psi_i\rangle_L\langle\Psi_i|_L)$$

and the final state $|\Psi_i\rangle_L\langle\Psi_i|_L$ is independent of $|\psi\rangle'_3$.

Replacement of this relation in (F.1) results in

$$|\psi\rangle_3\langle\psi|_3 \otimes \rho_L = \sum_i w_i(|\psi\rangle_3\langle\psi|_3 \otimes |\Psi_i\rangle_L\langle\Psi_i|_L) = |\psi\rangle_3\langle\psi|_3 \otimes \left(\sum_i w_i |\Psi_i\rangle_L\langle\Psi_i|_L\right).$$

□

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